Curtin
University of Technology

Numerical Modelling of Wave Propagation Using Higher Order Finite-Difference Formulas

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This thesis is presented for the Degree of
Doctor of Philosophy of Curtin University of Technology

May 2003

Abstract

A study has been conducted to investigate the numerical modelling of acoustic and elastic wave propagation using lower and higher order finite-difference representations of the wave equation.

Grid dispersion and edge reflection effects for acoustic wave propagation in two dimensional media were investigated. In addition, grid dispersion effects for elastic wave propagation in two dimensional media was carried out as well. In numerical models developed, the spreading loss of acoustic wave propagation for selected cases was accounted for. This numerical model was compared with that from published literature. A 2.5-D acoustic modelling approach used for a simulation of spherical spreading effects was developed. It was also applied to a geoacoustic model and results were compared with the field experimental data.

Initially, the study developed finite-difference formulas from 2^{nd} order up to 20^{th} order accuracies, for for $1st$ and 2^{nd} order derivatives and for 'half way' and 'one way' grid points, where Gauss elimination was used to derive a determinant of a matrix of Taylor expansions. The formulas have been evaluated by calculation of RMS errors. Exact solutions were used in estimating the errors for 4^{th}, 6^{th}, and up to 14^{th} order accuracies. The errors for 1^{st} order derivative have been examined by using analytic solutions. The formulas for 2^{nd} order up to 12^{th} order accuracies have been compared and checked with tables of finite-difference formulas from other published literature. The exact solution developed here is in good agreement with the analytic solution. Therefore, formulas derived are robust for use in the calculation.

The formulas for 2^{nd} order and 1^{st} order derivatives were applied for both homogeneous and heterogeneous formulations of acoustic models. The results indicated that 2^{nd} order accuracy gives less grid dispersion effects than a heterogeneous formulation which uses 1^{st} order derivatives. For higher order accuracy formulas, the 20^{th} order accuracy is not much different from 18^{th} and 16^{th} order accuracies. Grid points per wavelengths, p_x, equal to 15, 7.5 and 5 were adopted here. For p_x equal to 15, dispersion effects do not occur. Signals are slightly dispersed for p_x equal to 7.5, and significantly dispersed for p_x equal to 5. An elastic model using 1^{st} order derivatives of the finite-difference formulas with p_x equal to 7.5 and 5 yielded more dispersion effects than the comparable acoustic model. A 2-D acoustic modelling using a pseudospectral method was developed and compared also with the 20^{th} order accuracy finite-difference method. The pseudospectral method for p_x equal to 5 did not give less dispersion effects than the finite-difference method. Similar dispersion effects were, however, noticable in results derived using both the pseudospectral method for p_x equal to 3.75 and the finite-difference method. It was also found that the finite-difference method offer a faster computation than the pseudospectral method can provide.

In this study, various boundary conditions were developed notably the modified Reynolds and a combination of Reynolds, Cerjan and Dablain boundary conditions. It has been found that the Reynolds boundary condition gives less edge reflection effects than the Cerjan and Dablain boundary conditions. Even though Cerjan boundary condition and Dablain boundary condition were combined with Reynolds boundary condition or modified Reynolds boundary condition, they cannot offer less non-physical reflections. The Cerjan and Dablain boundary conditions require more grid points than do the Reynolds boundary. The modified Reynolds boundary conditions can give more accurate results than the standard Reynolds boundary condition. The modified Reynolds boundary conditions can reduce about 29% of the reflected amplitudes arising from use of Reynolds boundary condition. The modified Reynolds boundary condition was used in the present work in order to reduce non-physical reflection from array boundaries.

The scattered field calculated by the 2-D acoustic model and time domain finite-difference method has been compared with numerical examples from published literature calculated by a boundary element method, for a case involving acoustic wave propagation in an ocean environment covered by a half cylindrical ice floe, and where a half space is covered by a thick ice sheet with a half cylindrical ice keel. Results emerging from the present work correlated well with results presented in the published literature.

3-D algorithms are now available to model anisotropic structure, but these algorithms call for large computer memory and CPU time. This study has developed a new version of a numerical algorithm which makes it possible to simulate similar 3-D spreading effects in a 2-D environment, based on the finite-difference approximation of the 2.5-D acoustic wave equation for variable and constant density media. However, the configuration for 2.5-D acoustic wave propagation is achieved when the medium properties vary only in 2-D, and the source and receivers are located in the same plane. The method used to design the 2.5-D acoustic wave modelling is that a filter operator $F = K\sqrt{L}$ and damping coefficient C_c are applied to the 2-D wave equation to reduce propagating signal amplitude from the source temporally and spatially. Using the least errors of amplitudes of the 2.5-D modelling, the wavespeeds can be correlated linearly with combination of $K * C_c$. This result is a new contribution in 2.5-D acoustic modelling technique.

In this study, 2.5-D acoustic modelling was applied to a geoacoustic model of the southern Rottnest Basin and results were compared with the field experimental data. A representative airgun signal recorded in the near field was used as source pulse. Signals and spectrums of the signals from the field experiment are closely correlated with signals and spectrums of the signals from the model. However, some discrepancies were observable that might arise from a discrepancy between physical parameters adopted and real physical parameters of seabed sediments in the trial site, the southern Rottnest Basin.

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Chapter 1

Introduction

1.1 Research Background

Wave propagation theory is used by scientists and engineers, in particular acousticians and seismologists, in developing models for acoustic and elastic wave propagation. Usually acousticians are involved in developing the models in marine acoustics, of particular importance in naval operations, while seismologists develop the models which are of importance in geophysical exploration. In this document, the acoustic wave equation is used to describe acoustic wave propagation in fluid media, in which case only compressional waves are supported. Here, the elastic wave equation is used to describe elastic or seismic wave propagation in solid media, such that compressional and shear modes occur.

Over several decades, relating to the marine and geophysical environments, wave propagation modelling techniques have been developed increasingly by acousticians and seismologists in separate development paths, in order to find the best method for solving problems. Literature concerned with marine acoustics treats both high and low frequency modelling and the models often employ frequency domain approaches, even though many marine acoustic applications are centered on time domain analysis. This is due to the fact that in marine acoustics, acousticians often use narrow-band processing and only need to provide information about band-averaged energy distribution in space. Therefore, to analyze field experimental data in marine acoustics, commonly acousticians use a spectral analysis technique approach. Techniques developed for acoustic wave propagation modelling by this approach are also used for more general purposes, and can be used to cover large-scale environments.

The most important reason why acousticians choose spectral analysis techniques as an approach in marine acoustics is because the ocean is characterized by high temporal variability (Jensen, Kuperman, Porter and Schmidt, 1994). This causes strong and unpredictable signal fluctuations for long range propagation at traditional sonar frequencies.

Literature concerned with elastic modelling not only focuses particularly on wave propagation modelling of explosive sources with low frequency but also high frequency such as bore hole and sonic logging in complex solid media within geophysical geometries. To simulate elastic wave propagation in two or three-dimensional media, models use wide-band processing and the models often employ the time domain and numerical discretization at smaller scales and may also be more specialized than the ones for marine acoustics.

Apart from ray techniques at high frequency, Wave Number Integration Techniques, Normal Mode, and Parabolic Equation methods are the numerical solution techniques commonly applied for marine acoustic propagation modelling. These methods are based on the Helmholtz equation. The Helmholtz equation specifies a frequency in the solution of the wave equation. Hence the solution may involve a discrete time Fourier transform (DTFT). One or more of these approaches are still numerically efficient for the majority of forward modelling problems occurring in underwater acoustics, including propagation over very long range, with or without lateral variations in the environment (Jensen et al., 1994).

In marine acoustics, one common output of acoustic wave propagation modelling is a determination of the Transmission Loss (TL) as a function of range for a specific frequency. TL values can be combined with source Level (SL) values to calculate Sound Pressure level (SPL) for a frequency at given range, and at various depths in the water column (Penrose, 1996).

Despite the success of acoustic wave propagation models in explaining many of the observed phenomena, there are still a number of unresolved scientific issues, which can't be handled accurately by the traditional or classical numerical approach (Jensen et al., 1994). The methods used in marine acoustics is commonly treat only one-way propagation. Such methods are not able to handle complex problems such as discontinuous layer structures with a variety of characteristics, and consolidated or unconsolidated thin layer sediments involving a variety of materials. Such techniques are unable to treat backscattering problems directly.

Nevertheless, in recent years, backscatter has become of great importance in case of low and medium frequency scattering and reverberation from ocean boundaries. In relation to the importance of scattering and reverberation, low frequency active sonar systems have been increasingly developed, so it can be important to generalize various approaches, to allow modelling of some scattering and reverberation effects. Therefore there is an expectation to have models, which not only can be used to calculate Transmission Loss (TL) but also to use the models to interpret seabed environments involving discontinuous ranges and thin layered structures. One ensemble of numerical approaches, which are available for this purpose is based on some form of direct discretization of the governing equation. One such technique uses an explicit time domain finite-difference method. This method is more likely to be used by seismologists and geophysicists in the description of elastic wave propagation in solid media structures. Now, the method is also used in the underwater propagation literature (Jensen et al., 1994).

In numerical modelling, an explicit time-domain finite-difference method has advantages and disadvantages. One of the advantages of using the time-domain explicit finite-difference method is that the method needs less computer memory in contrast to the other methods like implicit finite-difference, pseudospectral, finite-element, and boundary-element methods.

The techniques that the implicit and explicit finite-difference methods use to calculate values for all grid points in the future time are different. While the implicit finite-difference method uses simultaneously all values of grid points in the present time to calculate those in future time, the explicit finite-difference method uses only some values of grid points, depending on the order of accuracy used, to derive only one value of the grid point at a time.

The pseudospectral method is a method that uses DTFT in calculating spatial derivatives of a relevant partial differential equation. Finite-element method adopted here is a method that uses global trial functions referred to as the

Galerkin method (Jensen et al., 1994). Results derived from the finite-element method were compared to those derived from the pseudospectral methods. Scattered field of transmission loss for the environment covered by ice floe and ice keel available in the literature that used boundary-element method was compared with scattered field derived here using the explicit finite-difference method.

The explicit finite-difference method is easier to code and to handle very complicated sediment properties and very complex environmental geometries than the implicit finite-difference method. One of the disadvantages of the explicit finite-difference method, for the same number of grid points per wavelength, is that the accuracy of this method depends on the order of accuracy of the finite-difference scheme. For lower order accuracy, the explicit time-domain finite-difference method is less accurate than the implicit finite-difference method. Therefore for modelling acoustic and elastic wave propagation using explicit time-domain finite-difference methods considerable effort has been focused on developing new solutions to increase accuracy. For example, some have a developed finite-difference formulas from low order to higher order accuracy (Keller and Pereyra, 1978; Fornberg, 1987). Some also have introduced new techniques like the staggered grid method (Madariaga, 1976; Aki and Richards, 1980; Virieux and Madariaga, 1982; Virieux, 1984; Levander, 1988). The concept of accuracy in the implementation of the finite-difference method is discussed below in Section 1.1.2.

Seismologists and geophysicists have made large contributions to developing numerical modelling of acoustic and elastic waves, using time domain explicit finite-difference methods. Some of them discuss grid dispersion and edge reflection issues in connection with the accurate modelling of acoustic and elastic waves (Kelly, Ward, Treitel and Alford, 1976; Alford, Kelly and Bore, 1974). Others introduced new solutions of boundary condition techniques to reduce non-physical reflections from array boundaries (Lindman, 1975; Reynolds, 1978; Clayton and Engquist, 1977; Cerjan, Kosloff, Kosloff and Reshef, 1985; Keys, 1985; Randall, 1988).

Methods to increase accuracy or minimize grid dispersion effects have also been introduced by (Madariaga, 1976; Aki and Richards, 1980; Virieux and

Madariaga, 1982; Virieux, 1984; Levander, 1988; Mittet, 1994). All these contributions have advantages and disadvantages in acoustic and elastic wave modelling, and in this study, they are used as theoretical background in developing numerical models.

Alford et al. (1974) have discussed an accuracy assessment of finite-difference modelling of the two dimensional(2-D) acoustic wave equation to compute signals at receivers located in the vicinity of a 90-degree edge embedded in an infinite acoustic medium. They show that grid coarseness is an important parameter affecting the accuracy of finite-difference methods, where the number of grid points per wavelength is a measure of grid coarseness. They found that to produce accurate results, ten or more grid points per wavelength, when second order accuracy finite-difference techniques are employed, and five grid points per wavelength at the frequency of the upper half-power point, when fourth order accuracy finite-difference techniques are employed, are needed. If an insufficient number of grid points per wavelength are used, grid dispersion effects lead to inaccurate results. In wave modelling, this is the first paper introducing and discussing the accuracy of the grid dispersion effect, called spatial dispersion, and in this case the acoustic model using a homogeneous formulation of the acoustic wave equation applied to an environmental geometry containing sharp corners. Here the homogeneous formulation of the acoustic wave equation is employed for propagation in an environment with constant density and various elastic properties, and the heterogeneous formulation for an environment with variable density and differing elastic properties.

After Alford et al. (1974), Kelly et al. (1976) introduced numerical modelling procedures using a finite-difference approach to the two-dimensional elastic wave equation. This approach can be used to produce synthetic seismograms for complex subsurface geometry and for arbitrary source-receiver separations. The finite-difference schemes are used for both homogeneous and heterogeneous formulations. For the homogeneous formulation assuming that density of the medium is constant, the standard boundary conditions between media of different elastic properties presented as Lamé parameters must be satisfied explicitly as compressional and shear velocities. For the heterogeneous formulation where the density of the medium is not constant, the boundary conditions and the elastic properties presented as Lamé parameters may be satisfied implicitly at each grid point of the finite-difference mesh.

It was concluded that the method of finite-differences provides the geophysical interpreter with synthetic seismograms for quite complex geologic models. The finite-difference techniques not only yield reliable arrival times for various seismic or elastic events, but also account for the variation in signal amplitude with subsurface elastic impedance contrasts and range. The accuracy of the results obtained is not only affected by grid coarseness, but also affected by boundary conditions. The boundary conditions also fail to handle reflection effects if the grid coarseness of the models are too small. The computational boundaries in the models i.e. non-physical boundaries, can product unwanted reflections. Therefore boundary condition formulations are also very important factors in the development of valid models.

Research shows that the two major problems in the numerical modelling of wave propagation using time domain finite-difference methods are non-physical reflections and grid dispersion. There is a significant number of methods that have been developed to minimize reflections from boundaries and reduce dispersion effects.

1.1.1 Non-physical reflections

To minimize or reduce non-physical reflections from computational array boundaries, a number of techniques have been developed, having advantages and disadvantages. Generally, these techniques can be divided into 3 categories: the first category using absorbing boundary conditions. This approach replaces the wave equation in a boundary region by one-way wave equations, which don't permit energy to propagate from the boundaries back into the numerical mesh. To reduce non-physical reflections, these methods don't need free spaces outside the specified boundaries. So these methods don't require significant additional amounts of CPU time and computational memory.

The second category uses non-reflecting boundary condition techniques. This approach is based on gradual reduction of the amplitudes in a strip of grid points along the boundary of the mesh. This technique is simpler and more robust than the first method, and can be applied to a wide variety of time dependent problems and the effectiveness of this boundary condition technique does not change for shallow angles of incidence. To minimize non-physical reflections gradually, the technique needs free spaces outside the specified boundaries. This in turn means that this technique needs more computational memory and CPU time than the first method.

The third case employs a transmissive sponge boundary condition technique. This technique applies a transmission operator and damping coefficient in such a way that waves are radiated in one direction and damped in the opposite, thereby attenuating any residual boundary reflection. This technique also need free spaces outside the specified boundaries and more computational memory and CPU time.

Absorbing boundary condition techniques

The first study of numerical boundary condition modelling techniques using absorbtion methods was introduced by Lysmerr and Kuhlemeyer (1969). They used this technique to analyze dynamic problems in elastic solid media, where a special viscous boundary condition is used to approximate a finite solid model system from an infinite one for analysis of steady state foundation vibration problems. A finite element method has been used in the numerical modelling. Lysmerr and Kuhlemeyer, and Smith (1974) studied a nonreflecting plane boundary for wave propagation to eliminate the non-physical reflections from boundaries. This exact or analytical formulation is independent of both frequency and incident angle and involves the superposition of solutions. But this method needs integration over the full computational mesh of $2n$-wave equation where n is the number of boundaries at which zero-reflection is desired. Lindman (1975) also studied a free-space boundary condition method for the time domain wave equation. This boundary condition was constructed by using projection operators. It was shown that the projection operator is not only used for one-dimensional waves but it can be extended for higher dimensions along with numerically efficient approximations to them as was described for higher-dimensional problems. In formulating an absorbing boundary condition, he was the first researcher to use correction functions and reflection coefficient curves as a function of angle of incident wave propagation.

Reynolds (1978) used a finite-difference method to introduce transparent boundary conditions for the numerical solution of wave propagation problems, which greatly reduces unwanted reflections from the edges due to the use of Dirichlet and Neumann boundary conditions. Reynolds compared the results of Dirichlet boundaries with the results of transparent boundaries. But the formulations of transparent boundaries give accurate results only for cases of one-dimensional modelling where the direction of wave propagation is perpendicular to the boundaries. Before Reynolds, for cases of two and three-dimensional acoustic and elastic modelling, Clayton and Engquist (1977) introduced an absorbing boundary condition method. In this method the boundary conditions are based on a technique of replacing or damping the wave equation that moves in to the boundary region, by one-way and two way wave equations, which don't permit energy to propagate from a boundary back into the numerical mesh. Like the transparent boundary condition, the absorbing boundary condition also still produces non-physical reflections from boundaries.

Keys (1985) discussed absorbing boundary conditions for acoustic media. The boundary conditions derived from his work are based on a somewhat different decomposition of the wave equation. The method is based on the boundary condition methods developed by Clayton and Engquist (1977), and Reynolds (1978). These boundary conditions use a simple physical interpretation of an absorbing boundary operator. The absorbing boundary operator is designed to absorb perfectly plane waves travelling in any two directions. The waves absorb plane waves according to the direction in which the waves are propagating. Nevertheless, using the method combining two or more absorption operators we can absorb any number of directions of the plane wave propagating. But these absorbing operators are still not effective in minimizing wave reflections from boundary, because the use of reflection coefficients depends on selected absorption directions over a range of incidence angles, and one and more incident angles must be defined before computing. If the wave front is curved or contains phases approaching the boundary from multiple directions, the method may be impractical.

A free-surface boundary condition for two-dimensional elastic finite-difference wave simulation has been introduced by Vidale and Clayton (1986). This method improved previous methods that were unstable for some physical ratios. This method is stable for both some physical ratios of shear and pressure velocities, and lateral heterogeneity. Randall (1988) introduced a new method of provision of absorbing boundary conditions to reduce reflected waves. The new absorbing boundary is efficient for two and three dimensional modelling in finite-difference calculations of acoustic and elastic wave propagation. In general use, this boundary condition is more absorptive than the second order paraxial-absorbing boundary condition developed by Clayton and Engquist (1977), and more numerically stable for any physical Poisson's ratio. This boundary condition method is based on Lindman's (1975) absorbing boundary condition method for scalar waves.

Long and Liow (1990) introduced a transparent boundary condition method for finite-difference elastic wave simulation. This method was based on the decomposition of an elastic wave into dilatation and rotational strains, and then the strain of wave equation was replaced at the grid boundary by the outgoing wave equation. Application of this transparent boundary condition is restricted to a medium that is homogeneous at the boundary to assure full separation of pressure waves from shear waves.

For practical purposes, absorbing transparent boundary methods are relatively successful because most researchers use these transparent and absorbing boundary formulas to minimize reflections from boundaries. But the reflections from the boundaries are still a problem as we attempt to increase the accuracy of the model.

Non-reflecting boundary condition techniques

Cerjan et al. (1985) introduced a nonreflecting boundary condition for discrete acoustic and elastic wave equations. This solution is used to avoid boundary effects in modelling acoustic and elastic wave propagation. In this method the numerical mesh is enlarged or added beside the outer sides of the boundaries. For constructing a finite-difference scheme the gradual absorption boundary method needs an alternative scheme. Using this method the amplitudes are gradually reduced in strip of grid points along the boundary of the mesh. This is a very simple method and can be applied to a wide variety of time dependent problems, in addition this method can be applied to Fourier methods for acoustic and elastic wave propagation. It is concluded that the method could be used after a slight modification with explicit finite-element or finite-difference techniques, and it is applied to a wide variety of acoustic and elastic problems, even including elastic calculations for models, which contain fluids and solids.

This method needs more computer memory than the previous method. This is because the non-reflecting boundary condition technique needs an extra numerical computation scheme to reduce the amplitude of waves before reflection from the boundary, than other methods like the transparent and absorbing boundary (Clayton and Engquist, 1977; Reynolds, 1978). But the effectiveness of this boundary condition does not decrease for shallow angles of incidence.

Transmissive sponge boundary condition techniques

Dablain (1986) proposed a transmissive sponge boundaries method as a simple method, similar to the non-reflecting boundary condition. This method used the transmission operator in such a way that it radiated in one direction and damped in the opposite, thereby attenuating any residual boundary conditions. This method also needs more free or extra grid space outside boundaries and consumes more computational memory and CPU time.

All of the boundary condition techniques discussed above can be used to absorb or reduce non-physical reflections of incident waves on the boundaries and give adequate results. But for practical application, it is still desirable to have a technique that is easy to code, efficient, effective and able to give more accurate results. There are some possibilities to modify and combine three types of the boundary condition techniques, and this research will be focused in part to study boundary condition techniques. Some of the boundary condition techniques more commonly used for practical application in numerical modelling of acoustic and elastic waves are the Reynolds boundary condition, the Cerjan boundary condition and the Dablain boundary condition. These techniques are studied in the present work by using lower order and higher order accuracies. A combination of the three boundary condition techniques is employed.

1.1.2 Grid dispersion effects

Many methods have been developed to reduce grid dispersion effects in time domain finite-difference wave propagation modelling. One method is by the increased accuracy from low to higher order. Other methods use quite different methods, such as staggered grid finite-difference and pseudo-spectral or Fourier transform methods. All of the methods aim to reduce grid dispersion effects, while employing the least possible computational grids per wavelength and CPU times.

Regular grid FD method

The conventional or regular grid finite-difference method is a method commonly used in finite-difference modelling. Alford et al. (1974) have introduced this method to study the accuracy of regular grid finite-difference of the two dimensional acoustic wave propagation modelling between 2^{nd} order accuracy and 4^{th} order accuracy. They suggest that for the same accuracy, using 2^{nd} order or lower order accuracy needed ten grid points per wavelength. Using 4^{th} order or higher order accuracy needed five grid points per wavelength.

The term 2^{nd} order accuracy is used here to present that the truncation error of a Taylor expansion of the finite-difference formulas is at 2^{nd} order. Similarly 4^{th} order accuracy represents to present that truncation error of Taylor expansion of the finite-difference formulas is in 4^{th} order. 8^{nd} order, 10^{nd} order up to 20^{nd} order accuracies can be used in this way.

Dablain (1986) introduced the application of higher order finite-difference schemes to the scalar wave equation and demonstrated that gains in computational efficiency can be achieved by using higher order approximations for the derivatives of the wave equation. He used a one-dimensional model to illustrate the relative accuracy of 2^{nd} and 4^{th} order in time and 2^{nd}, 4^{th} and 10^{th} order in space.

Fornberg (1988) introduced some tables of weights as a result of generation of finite-difference formulas on arbitrarily spaced grids up to the 4^{th} order derivative and up to 9^{th} order accuracy. The weights are coefficients of finite-difference formulas. The weights in the compact finite-difference formulas can be used for any order of derivative and to any order of accuracy on one-dimensional grids including the cases of one-sided and centered approximations at a grid point and at a 'half-way point' between grid points. Fornberg's (1988) paper is often used as a reference to develop wave propagation modelling using higher order regular and staggered grid finite-difference schemes. The accuracy of higher order finite-difference methods has been discussed. Wu, Lines and Lu (1996) conducted a study on the accuracy of the finite-difference operator from second order to eighth order for a scalar wave equation, by comparing numerical and analytical solutions quantitatively. They also presented an analysis of higher-order finite-difference schemes for application in 3-D full scalar wave equations. Wu et al. (1996) concluded that using 4^{th} or higher order accuracy in space for application of a finite-difference method in reverse-time migration could reduce CPU time and memory, especially for 3-D models. In 3-D models the use of the scheme with 4^{th} order accuracy in space can save more than 90% of memory and CPU time, when compared to that of 2^{nd} accuracy in space. This is because, 4^{th} order accuracy can use coarser grid spaces than 2^{nd} order accuracy.

Staggered grid method

Originally the staggered grid finite-difference method was introduced by Madariaga (1976) in his research on the dynamics of an expanding circular fault. In his research, Madariaga developed a staggered grid finite-difference scheme based on the 1^{st} order coupled elastic equations of motion and constitutive law in particle velocity and stress, which be used to model an expanding circular crack in an elastic space. Aki and Richards (1980) discussed and showed that staggered grid finite-difference schemes give more accurate results than regular grid finite-difference schemes. They give an example of an approximation of the regular and staggered grid central finite-difference scheme for an equation of motion, using 2^{nd} order accuracy, where the instability of the approximation of the regular grid scheme would grow with time. Also in their discussion, it is shown that the leading term of error of the grid of the finite-difference scheme constructed by using staggered grid finite-difference method is four times smaller than in the case of a regular grid finite-difference method. Another method has been proposed by Virieux and Madariaga (1982) to study dynamic faulting by using a finite-difference scheme. In his paper, Virieux (1984) represented staggered grid finite-difference scheme that includes both velocity and stress in the equation of motion. This staggered grid method is represented for modelling horizontal shear (SH) wave propagation in a generally heterogeneous medium. It solved the elastodynamic equations in term of velocity and stress.

A finite-difference method for modelling compressional (P) and vertical shear (SV) waves in heterogeneous media has also been presented by Virieux (1986). He shows that the stability condition and the P-wave phase velocity dispersion curve did not depend on Poisson's ratio. The S-wave phase velocity dispersion curve, on the other hand, was dependent on Poisson's ratio. Therefore, the same code used for elastic media, can be used for liquid media, where S-wave velocity goes to zero, and no special treatment was required for a liquid-solid interface. The method provided stable results for step discontinuities, as shown for a liquid layer above an elastic half-space media.

Properties of a 4^{th} and 2^{nd} order accuracy in space and 2^{nd} order accuracy in time of two-dimensional $P-SV$ staggered grid finite-difference schemes have been presented by Levander (1988). The finite-difference scheme is also based on the Madariaga-Virieux staggered-grid formulation. Their analysis for dispersion gave indications that the shortest wavelengths in the model need to be sampled at five grid points per wavelength for 4^{th} order accuracy and 10 grid points per wavelength for 2^{nd} order accuracy. The staggered grid scheme was easily written with higher order approximations to the spatial derivative, making it more efficient than other schemes for most modelling problems. It could also used to accurately simulate wave propagation in mixed acoustic-elastic media, making it ideal for modelling marine problems. Mittet (1994) uses a staggered grid finite-difference method, slightly different from the staggered grid method which has been used by Madariaga (1976).

In Mittet's (1994) work the stresses are located on reference grids, but the particle velocities or displacements are shifted half a grid step in the x and z direction relative to the reference grids. Mittet demonstrated how the spatial part of the boundary condition for an elastic field can be implemented numerically in a staggered coarse grid-modelling scheme with good accuracy, by using band-limited spatial delta functions and band limited first order derivatives of these spatial delta functions. Hestholm and Ruud (1994) included surface topography in their research on two-dimensional staggered grid FD elastic modelling. They concluded that the basic assumption of the calculations was that the topography function must be smooth, because if the function varies too greatly over a certain region, Gibb's phenomenon would arise and the resulting oscillations would destroy the simulation. Chen (1996) has developed a viscoacoustic numerical modelling algorithm tested and implemented for two dimensional media, using staggered grid finite-difference formulations. The numerical simulation of viscoacoustic responses was obtained by solving either two-dimensional homogeneous and heterogeneous isotropic linear viscoacoustic wave equations.

Staggered grid finite-difference was used by Graves (1996) to simulate seismic wave propagation in three-dimensional elastic media. He concluded that the flexibility and accuracy of the staggered-grid finite-difference algorithm make this technique a powerful tool in the analysis of wave propagation problems. Pitarka (1999) proposed a staggered grid with nonuniform spacing for three dimensional fourth order accuracy finite-difference modelling of seismic motions. From tests in this study, it was demonstrated that his proposed non-uniform staggered grid finite-difference formulation is efficient and sufficiently accurate in modelling wave propagation in three-dimensional elastic media. Saenger, Gold and Shapiro (2000) found that modelling elastic wave propagation with an explicit finite-difference scheme on a standard staggered grid might caused instability problems when the medium possesses high contrast discontinuities or strong heterogeneities. Therefore, they developed a new rotated staggered grid finite-difference operator for two and three dimensional elastic wave modelling. It is concluded that using such a modified grid it is possible to simulate the propagation of elastic waves in a medium containing cracks, pores or free surfaces, without applying boundary conditions, and this method did not cause instability problems.

Pseudospectral method

The pseudospectral method is a method which uses a discrete time Fourier transform (DTFT) to approximate derivatives of partial differential equation. Kreiss and Oliger (1972) originally proposed this method. Gazdag (1981) also studied modelling of the acoustic wave equation with Fourier transform or pseudospectral methods. He concluded that pseudospectral methods were considerably more accurate than finite-difference methods. But the transform method required more computation than the finite-difference method. Kosloff and Baysal (1982) used a pseudospectral method to test two problems, a single problem with a known analytic solution and a wedge problem. The latter case was also tested by physical modelling. The numerical results agreed with both the analytic and physical model results. Kosloff and Baysal conclude that the Fourier transform method requires fewer grid points than the finite-difference method to achieve the same solution. However, the Fourier transform method was more time consuming than the finite-difference method as computations were performed not locally around each element of the grid, but rather along complete lines in the coordinate direction.

Fornberg (1987) applied pseudospectral and finite-difference methods to the problem of two-dimensional elastic forward modelling. In his research, he stated that memory requirement and CPU time for each time step, using a pseudospectral method was less than second and fourth order finite-difference methods. Reshef, Kosloff, Edwards and Hsiung (1988) used the Fourier transform method to calculate three-dimensional acoustic modelling. Chen (1996) studied staggered-grid pseudospectral viscoacoustic wave simulation in two-dimensional media. His method is applied to solving either heterogeneous or homogeneous wave equations. Wu and Lees (1997) applied the pseudospectral method for free surface boundary conditions.

Wang (2001) presented a comparison of three-dimensional seismic wave simulation in anisotropic media using a staggered grid finite-difference method and a pseudospectral method. Wang's (2001) results indicated that when the staggered grid in higher order finite-difference operator was optimized based on Holberg operators, the higher order finite-difference method was faster than the pseudospectral method, even for a large scaled structure in three-dimensional models, with approximately the same accuracy.

From the studies discussed above, it logically follows that the best method to reduce or minimize grid dispersion effects of two and three dimensional acoustic and elastic modelling, is the pseudospectral method or Fourier transform method. For accuracy, this method is better than higher order accuracy finite-difference and staggered-grid finite-difference method. However regular grid higher-order finite-difference methods and staggered-grid finite-difference methods are relatively easy to code and need CPU times and require computer memory less than the pseudospectral method. Thus, most scientists and engineers use regular or staggered grid finite-difference methods in modelling acoustic and elastic wave propagation.

New methods to improve accuracy are developed continuously by looking for new techniques, or by increasing orders of accuracy of finite-difference formulas. In recent years, a few researchers have been interested in developing finite-difference formula algorithms, in order to improve accuracy by increasing orders of accuracy of finite-difference formulas. Fornberg (1987) has developed a technique to generate the tables of orders of the accuracy of higher order finite-difference formulas up to 8^{th} order, for either half way centre-difference scheme or one way centre-difference scheme, and also forward and backward-difference scheme. The motivation of his research was to apply regular grid finite-difference methods and staggered grid finite-difference methods in studying numerical acoustic and elastic modelling, and to study techniques to improve orders of the accuracy of finite-difference methods.

Because of the computational cost issues of time domain wave propagation modelling for realistic environments, researchers are motivated to employ gradual absorbtion techniques into 2-D numerical modelling to approximate a 3-D numerical modelling. The approach is called two and half dimensional (2.5-D) numerical modelling. This is applicable to wave propagation numerical modelling in a three-dimensional environment that has variations in two-dimension only. (Bleistein, 1986; Liner, 1991; Stockwell, 1995; Williamson and Pratt, 1995; Narayan, 1999).

Bleistein (1986) proposed a two and half dimensional (2.5-D) numerical modelling of plane wave propagation. He proposed an alternative method of constructing 2.5-D wave operators based on the high frequency asymptotic ray series (WKBJ) analysis. The objective of this work was to reduce the analysis of the in-plane wave propagation to a 2-D analysis, while retaining - at least asymptotically - the proper three-dimensional geometrical spreading. He has used it for the free space Green's function and for Kirchoff approximate upward scattered field from a single reflector. In both cases, he has concluded that the in-plane propagation of a wave in three dimensions can be described totally in terms of in-plane calculations. These are easier to generate than 3-D models. The idea is to ray-trace the wavefield in the $(x,\ z)$ - plane while allowing for out-of-plane spreading. In this way three-dimensional amplitude decay is honoured without three-dimensional ray tracing.

Liner (1991) introduced a theory of a 2.5-D acoustic wave equation for constant density media. This method is not only limited to ray tracing and a Green's function method as is the previous method. Liner considered the problem of deriving a 2.5-D wave equation. It is hoped that this will be useful for efficient, true amplitude finite-difference modelling in media. Stockwell (1995) also introduced a numerical modelling of 2.5-D wave equations using a high frequency asymptotic ray series (WKBJ). The results in high frequency analysis of Liner 2.5-D variable wave speed equation, indicate a maximum amplitude error ±35% in a linear model, where the wave speed doubles or halves from the beginning to the end of a ray-path. This method is better than Liner's method because it permits derivation of variable density (acoustic) 2.5-D wave operators.

Williamson and Pratt (1995) have given a brief critical review of important 2.5-D results to solve an alternative 2.5-D wave equation. The wave equation is generated from the 2-D wave equations by incorporating a filter for transforming 2.5-D arrivals to two dimensions (2-D). This filter is based on Bleistein's method for ray theoretical arrivals in general media. A significant result of this analysis is that there is little benefit in explicitly modelling either 2.5-D wave equation, because the wave field may be more simply produced by modelling the ordinary 2-D acoustic wave equation and then applying the relevant filter. Narayan (1999) presented the development of 2.5-D simulation techniques for acoustic wave propagation in media with variable density and velocity, using the techniques proposed by Stockwell (1995). Narayan discussed a comparative study of 2-D and 2.5-D responses damped spatially and temporally. He applied the model in a geometry without and with the inclusion of gas in an oil-bearing layer. His simulated wave propagation results showed that variation in density only affects the reflectivity of the layer. In this time domain modelling, the computational cost of 2.5-D modelling is $10 \sim 15\%$ more than the computational cost of the 2-D modelling. From this research, maximum and minimum amplitudes of 2-D and 2.5-D wave fronts are also presented for the constant and variable density cases. Limited conclusions are possible from these results regarding whether 2.5-D wave modelling satisfies geometrical spreading laws as does 3-D modelling. This means there remains a possibility to develop 2.5 D wave modelling which would give the best results and have strong correlation with a real three-dimensional environments.

The underwater acoustic propagation depends on various physical properties of seawater such as, among others, seawater acoustic absorption. Although the theory of the seawater acoustic absorption is well established (see, among others, Fisher and Simmons; Francois and Garrison; Francois and Garrison), it varies slightly depending on the local properties of the seawater. While frequency domain modelling of acoustic and elastic wave propagation incorporates the acoustic absorption coefficient, the time domain modelling using the finite-difference technique can not readily handle the absorption (Gerstoft, 1999). The technique developed in this study, however, incorporates the classical seawater acoustic absorption without molecular processes associated with Boric Acid ($B(OH)_3$) and Magnesium Sulphate ($MgSO_4$) in the model as presented in Chapter 8.

1.2 Research Objective

The main objective of this research is to develop lower order to higher order accuracy of time domain acoustic and elastic wave modelling, using an explicit finite-difference method. The specific objectives of the research are:

1. To develop 2-D and 2.5-D acoustic and elastic wave propagation and 3-D acoustic wave propagation modelling, for lower and higher order accuracy

including attenuation coefficients to account for spreading effects.

2. To develop visualization techniques to show acoustic and elastic wave propagation for practical and educational purposes, that can give an understanding of underwater acoustic and elastic wave propagation.

3. To apply and combine two types of boundary conditions that have been developed for explicit finite-difference modelling.

4. To compare results of the models with results from other research.

5. To study grid dispersion effects from finite-difference schemes, problems associated with non-physical reflections from boundaries for every order of accuracy from 2^{nd} order up to 20^{th} order, transmission loss associate with beam spreading effects and damping coefficients of modelling 2.5-D acoustic and elastic waves.

6. To study signals of acoustic wave propagation from field experimental data.

1.3 Aims

The main aim of this research is to develop a numerical model of acoustic and elastic waves, that can be used to visualize wave propagation, and predict the transmission loss associated with beam spreading effects of acoustic propagation in shallow water, involving a variety of geo-acoustic conditions. Development of such a model is critical in studying seabed sediments involving discontinuities in sediment layer properties, including thin layers. This study will also contribute and provide a facility suitable for educational purposes, concerning the propagation of acoustic waves in shallow underwater environments.

1.4 Outline

The studies reported here are concentrated on developing: numerical algorithms of acoustic and elastic waves in heterogeneous and complex structures, using time domain explicit finite-difference methods, methods to visualize wave fronts, and the time domain representation of signals. The thesis is divided into nine chapters.

Chapter 1 describes the background, the objectives and the contribution of the research, associated with time domain numerical modelling of acoustic and elastic waves, using explicit finite-difference methods.

Chapter 2 reviews acoustic and elastic wave modelling techniques, using explicit finite-difference methods and time domain solutions to visualize wave front propagation, in shallow-water for homogeneous and heterogeneous media. It reviews boundary condition derivative techniques for acoustic and elastic wave modelling and reviews techniques to calculate transmission loss arising from geometrical spreading of the acoustic and elastic wave propagation.

Chapter 3 presents the first contribution of the thesis, where a Taylor series is used to generate finite-difference formulas for low order and higher orders up to 16^{th} order accuracy. In the numerical modelling of acoustic and elastic wave equations in the time domain, these formulas are very important for the design of finite-difference schemes.

Chapter 4 presents the second contribution of this thesis, where the grid dispersion effects of time domain numerical modelling of acoustic and elastic waves are studied by using high order accuracy of finite-difference formulas from 2^{nd} to 20^{th}-order accuracy.

Chapter 5 presents the third contribution of this thesis, where Reynolds transparent boundary and modification of the Reynolds boundary methods are presented. Non-physical reflections from the radiation boundaries of numerical schemes of acoustic and elastic waves for several boundary conditions method are compared and analyzed, to obtain the capability of each method in decreasing non-physical reflection from boundaries.

Chapter 6 presents the fourth contribution of this thesis, where the capability of the model in the calculation of transmission loss (TL) due to spreading is studied, by comparing results of the algorithm with results of published literature using the same environmental model.

Chapter 7 presents the fifth contribution of the thesis, where a filter operator F and damping coefficient C_d for 2.5-D acoustic modelling are studied in order to obtain the same spreading effects as provided by 3-D acoustic wave modelling.

Chapter 8 presents the last contribution of this thesis, where 2.5-D acoustic wave propagation is simulated for comparison with field experimental data. In this study an attenuation function is involved into the acoustic algorithm to present acoustic wave propagation in the real physical environment related to the experimental program.

Chapter 9 offers conclusions and recommendations for future research work.

Chapter 2

Review of numerical modelling of acoustic and elastic waves

This chapter discusses visualization techniques and prediction of wave propagation in under-water environments using finite-difference methods and time domain solutions for homogeneous and heterogeneous media. Homogeneous media are media characterised by only one phase of its physical properties. Heterogeneous media are media characterised by two or more physical properties phases. The boundary condition derivative techniques for acoustic and elastic wave modelling are also discussed to calculate transmission loss of acoustic and elastic wave propagation. The review of wave modelling techniques in this chapter is collected and summarized from journals and books for the reader's convenience and establishes the foundations for the development of the research. The purpose of this review is to direct selection of the most suitable methods, which can be used in the research.

2.1 Acoustic wave equation

Commonly the acoustic wave equation is used to describe acoustic wave propagation in fluid media. In this case the acoustic wave equation describes compressional waves only. Acoustic wave equations are derived in many ways (Brekhovskikh, 1960; Keiswetter, Black and Schmeissner, 1996; Morse and Ingard, 1968). The equations of the acoustic field in an heterogenous medium can be derived using

Euler's relation and the equation of continuity, which have the following forms,

$$\frac{\partial p}{\partial t} + \rho . C_o^2 . \left[\vec{\nabla} \bullet \vec{v} \right] = 0 \qquad \text{Euler's equation} \tag{2.1}$$

$$\frac{1}{\rho} \left[\vec{\nabla} p \right] + \frac{\partial \vec{v}}{\partial t} = 0 \qquad \text{Equation of continuity} \tag{2.2}$$

Where p is the acoustic pressure, $\vec{v}$ is the particle velocity in the wave, ρ is the density, and C_o is the compressional velocity of sound in the media. By eliminating the time derivative of equation (2.1) and the divergence of equation (2.2), the equation (2.1) and (2.2) can be changed to be the wave equation. The time derivative of Euler's equation (2.1) is,

$$\frac{\partial}{\partial t} \left\{ \frac{\partial p}{\partial t} + \rho . C_o^2 \left[\vec{\nabla} \bullet \vec{v} \right] \right\} = 0$$

$$\frac{\partial^2 p}{\partial t^2} + \rho . C_o^2 \frac{\left[\vec{\nabla} \bullet \vec{v} \right]}{\partial t} = 0 \tag{2.3}$$

And the divergence of the continuum equation (2.2) is

$$\vec{\nabla} \bullet \left\{ \frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho} \left[\vec{\nabla} p \right] \right\} = 0$$

$$\frac{\partial \left[\vec{\nabla} \bullet \vec{v} \right]}{\partial t} + \vec{\nabla} \bullet \left[\frac{1}{\rho} \vec{\nabla} p \right] = 0 \tag{2.4}$$

Solving the equation (2.3) for $\frac{\partial [\vec{\nabla} \bullet \vec{v}]}{\partial t}$ where,

$$\frac{\partial [\vec{\nabla} \bullet \vec{v}]}{\partial t} = - \vec{\nabla} \bullet \left[\frac{1}{\rho} \vec{\nabla} p \right] \tag{2.5}$$

And substituting equation (2.5) into the equation (2.4) yields,

$$\frac{\partial^2 p}{\partial t^2} + \rho . C_o^2 \left\{ - \vec{\nabla} \bullet \left[\frac{1}{\rho} \vec{\nabla} p \right] \right\} = \delta(r) . f(t) \tag{2.6}$$

Equation (2.6) is a scalar equation that is represented as the acoustic wave equation in heterogeneous media. Here, $\delta(r)$ is the Dirac delta and $f(t)$ is the source function. The Dirac delta $\delta(r)$ is used to position the source in space, and the source function $f(t)$ is defined as a characteristic of the shape in time. Commonly the acoustic wave equation is given in terms of pressures. But the acoustic wave equation can also be described in terms of the acoustic velocity potential of the particle of the fluid. The equation (2.6) can be rewritten in three-dimensional form as follows,

$$\frac{\partial^2 p}{\partial t^2} - \rho.C_o^2.\left[\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right)\right] = \delta(r).f(t) \qquad (2.7)$$

In a medium, where physical properties are independent of direction. Equation (2.7) simply becomes

$$\frac{\partial^2 p}{\partial t^2} - C_o^2.\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right] = \delta(r).f(t) \qquad (2.8)$$

Equation (2.8) is an acoustic wave equation for homogeneous media. The acoustic wave equation can also be presented as a governing equation as follows (Alford et al., 1974);

$$\left[\nabla^2 - \frac{1}{C_o^2}\frac{\partial^2}{\partial t^2}\right].u\,(\rho, \phi, t) = -4.\pi.\frac{\delta(\rho - \rho_s)\delta(\phi - \phi_s)f(t)}{\rho} \qquad (2.9)$$

This equation describes the acoustic velocity potential $u(\rho, \phi, t)$ in a homogeneous media with a line source distribution, located (ρ_s, ϕ_s) with respect to the origin of the cylindrical coordinate system and in the time domain. The equation (2.9) can also be written in two-dimensional form,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{C_o^2}\frac{\partial^2}{\partial t^2}\right].u(\rho, \phi, t) = -4.\pi.\frac{\delta(\rho - \rho_s)\delta(\phi - \phi_s)f(t)}{\rho} \qquad (2.10)$$

Where C_o is the velocity of the acoustic propagation in the medium, ∇^2 is the Laplacian operator, $f(t)$ is the time variation of the source distribution, and $-4.\pi.\frac{\delta(\rho-\rho_s)\delta(\phi-\phi_s)f(t)}{\rho}$ is the normalized two-dimensional Dirac delta function at the source location.

2.2 Elastic wave equation

Commonly the elastic wave equation is used to describe propagation of elastic waves or seismic waves in solid media. Elastic or seismic wave propagation involves both compressional and shear deformations. The elastic or seismic wave equation is derived from the equation of motion. In general, the theory of the mechanics of a continuum are derived from the static condition of equilibrium, using d'Alembert's principle (Nedoma, 1998). The equation of motion can be given as;

$$\frac{\partial \tau_{ij}(x,t)}{\partial x_j} + \rho(x,t).F_i(x,t) = \rho(x,t)\frac{\partial^2 u_i(x,t)}{\partial t^2} + \alpha(x)\frac{\partial u_i(x,t)}{\partial t} \qquad (2.11)$$

Where $i = 1, \cdots, N$, $j = 1, \cdots, N$, $N = 2$ and 3, x_j is function of the j-th component of the coordinates, τ_{ij} is the ij-th components of the stress, $\rho = \rho(x,t)$ is the density of the medium, $u = u_i$ is the displacement, F_i is the i-th component of the body forces per unit volume, $\partial^2 u_i(x,t)/\partial t^2$ is the i-th component of acceleration, $\rho\partial^2 u_i(x,t)/\partial t^2$ is the i-th component of the inertia force and $\partial u_i(x,t)/\partial t$ is the i-th component of the velocity of the considered volume element, and $\alpha = \alpha(x)$ presents the coefficient of elastic wave damping. The medium is considered as heterogeneous and it is assumed that elastic coefficients and the density are functions of spatial coordinates only. It is assumed that inhomogeneous absorption (damping) is defined by the coefficient of absorption (damping) α in the medium. If it is assumed that the linear source of elastic waves is parallel with the axis $x_2 \equiv y$ and all the physical parameters are independent on this coordinate, equation (2.11) then yields,

$$\frac{\partial \tau_{11}(x,t)}{\partial x_1} + \frac{\partial \tau_{13}(x,t)}{\partial x_3} + \rho(x)F_1(x,t) = \rho(x)\frac{\partial^2 u_1(x,t)}{\partial t^2} + \alpha(x)\frac{\partial u_1(x,t)}{\partial t} \tag{2.12}$$

$$\frac{\partial \tau_{21}(x,t)}{\partial x_1} + \frac{\partial \tau_{23}(x,t)}{\partial x_3} + \rho(x)F_2(x,t) = \rho(x)\frac{\partial^2 u_2(x,t)}{\partial t^2} + \alpha(x)\frac{\partial u_2(x,t)}{\partial t} \tag{2.13}$$

$$\frac{\partial \tau_{31}(x,t)}{\partial x_1} + \frac{\partial \tau_{33}(x,t)}{\partial x_3} + \rho(x)F_3(x,t) = \rho(x)\frac{\partial^2 u_3(x,t)}{\partial t^2} + \alpha(x)\frac{\partial u_3(x,t)}{\partial t} \tag{2.14}$$

Hooke's law is defined as,

$$\begin{aligned}
\tau_{11}(x,t) &= (\lambda(x) + 2.\mu(x))\frac{\partial u_1(x,t)}{\partial x_1} + \lambda(x)\frac{\partial u_3(x,t)}{\partial x_3} \\
\tau_{33}(x,t) &= (\lambda(x) + 2.\mu(x))\frac{\partial u_3(x,t)}{\partial x_3} + \lambda(x)\frac{\partial u13(x,t)}{\partial x_1} \\
\tau_{13}(x,t) = \tau_{31}(x,t) &= \mu(x)(\frac{\partial u_3(x,t)}{\partial x_3} + \frac{\partial u_1(x,t)}{\partial x_1}) \\
\tau_{12}(x,t) = \tau_{21}(x,t) &= \mu(x)\left(\frac{\partial u_2(x,t)}{\partial x_1}\right) \\
\tau_{23}(x,t) = \tau_{32}(x,t) &= \mu(x)\left(\frac{\partial u_2(x,t)}{\partial x_3}\right)
\end{aligned}$$

Where λ and μ are Lamé's coefficients for elastic isotropic medium. The elastic coefficients λ and μ, and the density ρ are independent on one-coordinate x_2 and time t. Then equation (2.12) yields,

$$\begin{aligned}
\rho(x)\frac{\partial^2 u_2(x,t)}{\partial t^2} + \alpha(x)\frac{\partial u_2(x,t)}{\partial t} =& \frac{\partial}{\partial x_1}\left\{\mu(x)\frac{\partial u_2(x,t)}{\partial x_1}\right\} \\
&+ \frac{\partial}{\partial x_3}\left\{\mu(x)\frac{\partial u_2(x,t)}{\partial x_3}\right\} + \rho(x)F_2(x,t)
\end{aligned} \tag{2.15}$$

$$\begin{aligned}\rho(x)\frac{\partial^2 u_1(x,t)}{\partial t^2}+\alpha(x)\frac{\partial u_1(x,t)}{\partial t} =&\frac{\partial}{\partial x_1}\left\{(\alpha(x)+2.\mu(x))\frac{\partial u_1(x,t)}{\partial x_1}+\alpha(x)\frac{\partial_3(x,t)}{\partial x_3}\right\}\\ &+\frac{\partial}{\partial x_3}\left\{\mu(x)\left[\frac{\partial u_1(x,t)}{\partial x_3}+\frac{\partial u_3(x,t)}{\partial x_1}\right]\right\}\\ &+\rho(x)F_1(x,t)\end{aligned} \tag{2.16}$$

$$\begin{aligned}\rho(x)\frac{\partial^2 u_3(x,t)}{\partial t^2}+\alpha(x)\frac{\partial u_3(x,t)}{\partial t} =&\frac{\partial}{\partial x_1}\left\{\mu(x)\left[\frac{\partial u_1(x,t)}{\partial x_3}+\frac{\partial u_3(x,t)}{\partial x_1}\right]\right\}+\\ &\frac{\partial}{\partial x_3}\left\{(\alpha(x)+2.\mu(x))\frac{\partial u_3(x,t)}{\partial x_3}+\alpha(x)\frac{\partial_1(x,t)}{\partial x_1}\right\}\\ &+\rho(x)F_3(x,t)\end{aligned} \tag{2.17}$$

The equations present the system of equations of motion for pressure P and shear vertical SV waves, and these equations also satisfy shear horizontal SH waves. In x, y, z coordinate system the equations can be presented as,

$$\frac{\partial}{\partial v}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+\rho F_y=\rho\frac{\partial^2 v}{\partial t^2}+\alpha\frac{\partial v}{\partial t} \tag{2.18}$$

$$\frac{\partial}{\partial x}\left\{(\lambda+2.\mu)\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial z}\right\}+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right\}+\rho F_x=\rho\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t} \tag{2.19}$$

$$\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right\}+\frac{\partial}{\partial z}\left\{(\lambda+2.\mu)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x}\right\}+\rho F_z=\rho\frac{\partial^2 w}{\partial t^2}+\alpha\frac{\partial w}{\partial t} \tag{2.20}$$

Where: $v=v(x,y,z,t)$, $u=u(x,y,z,t)$, $w=w(x,y,z,t)$ are the displacements. $\rho=\rho(x,y,z)$ is the density of the medium, $\mu(x,y,z)$ and $\lambda(x,y,z)$ are the Lame's elastic coefficients, and $\alpha=\alpha(x,y,z)$ is the coefficient of absorption (damping). These equations are written as homogeneous and heterogeneous formulations of the elastic wave equation in an isotropic medium, which are used to describe the compressional and shear waves propagation in elastic medium (Kelly et al., 1976).

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \tag{2.21}$$

$$\rho\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \tag{2.22}$$

Equations (2.22) and (2.21) are the heterogeneous formulation of the elastic wave equation without absorption or damping coefficients. The equations can be derived as follows;

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[(\lambda+2\mu)\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right]-\frac{\partial}{\partial x}\left[2\mu\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \tag{2.23}$$

$$\rho\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial z}\left[(\lambda+\mu)\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\right)\right]-\frac{\partial}{\partial z}\left[2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \quad (2.24)$$

If $\alpha=\sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the compressional wave velocity and $\beta=\sqrt{\frac{\mu}{\rho}}$ is the shear wave velocity. The equation above and the equation for the z-axis component can be written as,

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[\alpha^2\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right]-\frac{\partial}{\partial x}\left[2\beta^2\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial z}\left[\beta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \quad (2.25)$$

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial z}\left[\alpha^2\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\right)\right]-\frac{\partial}{\partial z}\left[2\beta^2\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[\beta^2\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \quad (2.26)$$

If they are reduced or it is assumed that ρ, λ, and μ are constant for particular medium, the equations may be presented as,

$$\rho\frac{\partial^2 u}{\partial t}=(\lambda+2\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial w}{\partial x\partial z}\right)$$

$$\frac{\partial^2 u}{\partial t}=\frac{(\lambda+2\mu)}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial w}{\partial x\partial z}\right)$$

$$\frac{\partial^2 u}{\partial t}=\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\beta^2\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial w}{\partial x\partial z}\right) \quad (2.27)$$

$$\frac{\partial^2 w}{\partial t}=\alpha^2\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 u}{\partial x\partial z}\right)+\beta^2\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial u}{\partial x\partial z}\right) \quad (2.28)$$

Equations (2.27) and (2.28) are homogeneous formulations of the elastic wave equation.

2.3 Finite-difference Formulas

Generally, linear spatial differential equations of acoustic and elastic waves are used by acousticians and seismologists to model propagation of the acoustic and elastic waves in underwater and under seabed sediments. For solving the 1^{st} and the 2^{nd} order derivatives of the wave equations, finite-difference is one of several methods using discrete approximations that can be used to calculate linear combination of approximation of the function, at grid points. The accuracy of the approximation depends on order of accuracy used to calculate the model. The number of grid points whose corresponding values are used to derive a new value in a particular grid point, in a finite-difference scheme depends on the accuracy level developed. Finite-difference weights of the grid points are very important to calculate approximate values in finite-difference method. Finite-difference weights are more widely known as finite-difference formulas. The finite-difference formulas are commonly derived from a Taylor series.

Several studies have produced techniques to generate finite-difference formulas (Keller and Pereyra, 1978; Fornberg, 1988). Keller and Pereyra (1978) presented tables of coefficients for higher order of accuracy of a compact difference scheme, using the following definition; the least number of grid points can be used to obtain higher order accuracy approximation to the first 10 derivatives of smooth functions on uniform grids. In the paper, Keller and Pereyra discussed the generation of finite-difference formulas on arbitrary space grids. They produced a table of the finite-difference formulas, for 2^{nd} order of accuracy (h^2) up to 10^{th} order of accuracy (h^{10}), and for 2^{nd} order derivative up to 10^{th} order derivative. In this study, to derive the formulas, a symbolic manipulation method was used.

Fornberg (1988) presented a study of two simple recursion relations to generate the weights for any order of derivative, including the 0^{th} derivative for interpolation. They included the cases of one-side and centre approximations at a grid point and a halfway point between grid points. These researches have produced compact tables of finite-difference formulas. The formulas are very useful and may be used directly to design numerical models of acoustic or elastic wave equations. Tables similar to those provided by Keller and Pereyra and Fornberg are generated here for up to the 20^{th} order accuracy. Results generated here for the 2^{nd} order up to the 10^{th} order accuracy are compared with those from Keller and Pereyra and Fornberg and they are in very good agreement.

Such finite-difference formulas are needed by engineers and scientists to design finite-difference schemes for numerical modelling. Some studies apply the finite-difference formulas into a regular finite-difference scheme for modelling acoustic and elastic waves (Alford et al., 1974; Kelly et al., 1976; Dablain, 1986). Others apply the finite-difference formulas into a staggered grid finite-difference scheme (Aki and Richards, 1980; Levander, 1988; Mittet, 1994; Hestholm and Ruud, 1994; Keiswetter et al., 1996; Chen, 1996). In finite-difference methods, commonly, regular and staggered grid techniques are used in the calculation of numerical approximation of the models. Generally, regular techniques are implemented in acoustic and elastic wave modelling for practical purposes. Staggered grid techniques are usually implemented to produce more accurate results in wave modelling, for the equation which has lower order in its spatial derivative, like the heterogeneous formulation of the acoustic and elastic wave equation. From previous equations, the description for the heterogeneous formula of the acoustic wave equation in the two-dimensional media without a source, can be given as,

$$\frac{\partial^2 p}{\partial t^2} = \rho C_o^2 \left[\frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial p}{\partial z} \right) \right] \tag{2.29}$$

The second derivative with respect to time can be approximated using a standard central difference equation as,

$$\frac{\partial^2 p}{\partial t^2} = \frac{p_{n,m}^{k+1} - 2.p_{n,m}^{k} + p_{n,m}^{k-1}}{(\Delta t)^2} \tag{2.30}$$

Where $p_{n,m}^k = p(n\Delta x, m\Delta z, k\Delta t)$, Δx, and Δz represent the distance between the horizontal and vertical grid points, respectively, and Δt is the time step increment. The spatial derivatives can also be approximated by difference equation such as,

$$\left\{ \frac{\partial}{\partial x} \left[\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right) \right] \right\}_{n,m}^{k} = \frac{\rho_{n+\frac{1}{2},m}^{-1}}{\Delta x} \left[\frac{p_{n+1,m}^k - p_{n,m}^k}{\Delta x} \right] - \frac{\rho_{n-\frac{1}{2},m}^{-1}}{\Delta x} \left[\frac{p_{n,m}^k - p_{n-1,m}^k}{\Delta x} \right] \tag{2.31}$$

This equation is for the x component of motion. The equation for the z component can be derived similarly:

$$\left\{ \frac{\partial}{\partial z} \left[\frac{1}{\rho} \left(\frac{\partial p}{\partial z} \right) \right] \right\}_{n,m}^{k} = \frac{\rho_{n,m+\frac{1}{2}}^{-1}}{\Delta z} \left[\frac{p_{n,m+1}^k - p_{n,m}^k}{\Delta z} \right] - \frac{\rho_{n,m-\frac{1}{2}}^{-1}}{\Delta z} \left[\frac{p_{n,m}^k - p_{n,m-1}^k}{\Delta z} \right] \tag{2.32}$$

Based on equation (2.8), the homogeneous form of two-dimensional acoustic wave equation can be presented as

$$\frac{\partial^2 p}{\partial t^2} - \rho C_o^2 \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} \right] = \delta(r) f(t) \tag{2.33}$$

A finite-difference formulation of Equation (2.33) can be approximated in rectangular coordinates by the explicit second order difference scheme (Alford et al., 1974) as,

$$\begin{aligned} p_{n,m}^{k+1} =& 2(1-\gamma^2) p_{n,m}^k + \gamma^2 (p_{n+1,m}^k + p_{n-1,m}^k + p_{n,m+1}^k + p_{n,m-1}^k) + \\ & p_{n,m}^{k-1} + O(h^2 + (\Delta t)^2) \end{aligned} \tag{2.34}$$

Here $\Delta x = \Delta z = h$ is the grid size in the x and z directions, respectively; Δt is the time step; n, m, k are integers such that $x = n\Delta x$, $z = m\Delta z$, $t = k\Delta t$; $\gamma = \frac{C_o \Delta t}{h}$ and $O(h^2)$ indicates the scheme approximates the corresponding partial differential equation to order h^2 . An alternative expression may be obtained by using the more accurate fourth order as follows,

$$\begin{aligned} p^k_{n,m} =& (2-5\gamma^2)p^k_{n,m} + \frac{\gamma^2}{12}\Big\{16\left[p^k_{n+1,m}+p^k_{n,m+1}+p^k_{n-1,m}+p^k_{n,m-1}\right] - \\ & \left[p^k_{n+2,m}+p^k_{n,m+2}+p^k_{n-2,m}+p^k_{n,m-2}\right]\Big\} - p^{k-1}_{n,m} + O(h^4,(\Delta t)^2) \end{aligned} \tag{2.35}$$

For the heterogeneous formulation of the elastic wave equation, the two dimensional form of finite-difference scheme can be applied as follows (Kelly et al., 1976),

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\alpha^2\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\right] - \frac{\partial}{\partial x}\left[2\beta\frac{\partial w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\beta^2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \tag{2.36}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u^{k+1}_{n,m} - 2u^k_{n,m} + u^{k-1}_{n,m}}{(\Delta t)^2} \tag{2.37}$$

$$\begin{aligned} \frac{\partial}{\partial x}\left[\alpha^2\left(\frac{\partial u}{\partial x}\right)\right] =& \frac{1}{\Delta x}\Bigg\{\left[\frac{(\alpha_{n+1,m})^2+(\alpha_{n,m})^2}{2}\right]\left[\frac{u^k_{n+1,m}-u^k_{n,m}}{\Delta x}\right] - \\ & \left[\frac{(\alpha_{n,m})^2+(\alpha_{n-1,m})^2}{2}\right]\left[\frac{u^k_{n,m}-u^k_{n-1,m}}{\Delta x}\right]\Bigg\} \end{aligned} \tag{2.38}$$

$$\begin{aligned} \frac{\partial}{\partial x}\left[\alpha^2\left(\frac{\partial w}{\partial z}\right)\right] =& \frac{1}{2\Delta x}\Bigg\{(\alpha_{n+1,m})^2\left[\frac{w^k_{n+1,m+1}-w^k_{n+1,m-1}}{\Delta z}\right] - \\ & (\alpha_{n-1,m})^2\left[\frac{w^k_{n-1,m+1}-w^k_{n-1,m-1}}{\Delta z}\right]\Bigg\} \end{aligned} \tag{2.39}$$

$$\begin{aligned} \frac{\partial}{\partial x}\left[2\beta^2\left(\frac{\partial w}{\partial z}\right)\right] =& \frac{2}{2\Delta x}\Bigg\{(\beta_{n+1,m})^2\left[\frac{w^k_{n+1,m+1}-w^k_{n+1,m-1}}{2\Delta z}\right] - \\ & (\beta_{n-1,m})^2\left[\frac{w^k_{n-1,m+1}-w^k_{n-1,m-1}}{2\Delta z}\right]\Bigg\} \end{aligned} \tag{2.40}$$

$$\begin{aligned} \frac{\partial}{\partial z}\left[\beta^2\left(\frac{\partial u}{\partial z}\right)\right] =& \frac{1}{\Delta z}\Bigg\{\left[\frac{(\beta_{n,m+1})^2+(\beta_{n,m})^2}{2}\right]\left[\frac{u^k_{n,m+1}-u^k_{n,m}}{\Delta z}\right] - \\ & \left[\frac{(\beta_{n,m})^2+(\beta_{n,m-1})^2}{2}\right]\left[\frac{u^k_{n,m}-u^k_{n,m-1}}{\Delta z}\right]\Bigg\} \end{aligned} \tag{2.41}$$

$$\frac{\partial}{\partial z}\left[\beta^2\left(\frac{\partial w}{\partial x}\right)\right] = \frac{1}{2\Delta z}\left\{(\beta_{n,m+1})^2\left[\frac{w^k_{n+1,m+1} - w^k_{n-1,m+1}}{\Delta x}\right] - (\beta_{n,m-1})^2\left[\frac{w^k_{n+1,m-1} - w^k_{n-1,m-1}}{\Delta x}\right]\right\} \tag{2.42}$$

These are finite-difference schemes of elastic wave equation for x-component of motion. In the same way, the z-component of motion may be approximated as,

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z}\left[\alpha^2\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right] - \frac{\partial}{\partial z}\left[2\beta^2\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\beta^2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \tag{2.43}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{w^{k+1}_{n,m} - 2w^k_{n,m} + w^{k-1}_{n,m}}{(\Delta t)^2} \tag{2.44}$$

$$\frac{\partial}{\partial z}\left[\alpha^2\left(\frac{\partial w}{\partial z}\right)\right] = \frac{1}{\Delta z}\left\{\left[\frac{(\alpha_{n,m+1})^2 + (\alpha_{n,m})^2}{2}\right]\left[\frac{u^k_{n,m+1} - u^k_{n,m}}{\Delta z}\right] - \left[\frac{(\alpha_{n,m})^2 + (\alpha_{n,m-1})^2}{2}\right]\left[\frac{u^k_{n,m} - u^k_{n,m-1}}{\Delta z}\right]\right\} \tag{2.45}$$

$$\frac{\partial}{\partial z}\left[\alpha^2\left(\frac{\partial u}{\partial x}\right)\right] = \frac{1}{2\Delta z}\left\{(\alpha_{n,m+1})^2\left[\frac{u^k_{n+1,m+1} - u^k_{n-1,m+1}}{\Delta x}\right] - (\alpha_{n,m-1})^2\left[\frac{u^k_{n+1,m-1} - u^k_{n-1,m-1}}{\Delta x}\right]\right\} \tag{2.46}$$

$$\frac{\partial}{\partial z}\left[2\beta^2\left(\frac{\partial u}{\partial x}\right)\right] = \frac{2}{2\Delta z}\left\{(\beta_{n,m+1})^2\left[\frac{u^k_{n+1,m+1} - u^k_{n-1,m+1}}{2\Delta x}\right] - (\beta_{n,m-1})^2\left[\frac{u^k_{n+1,m-1} - u^k_{n-1,m-1}}{2\Delta x}\right]\right\} \tag{2.47}$$

$$\frac{\partial}{\partial x}\left[\beta^2\left(\frac{\partial w}{\partial x}\right)\right] = \frac{1}{\Delta x}\left\{\left[\frac{(\beta_{n+1,m})^2 + (\beta_{n,m})^2}{2}\right]\left[\frac{w^k_{n+1,m} - w^k_{n,m}}{\Delta x}\right] - \left[\frac{(\beta_{n,m})^2 + (\beta_{n-1,m})^2}{2}\right]\left[\frac{w^k_{n,m} - w^k_{n-1,m}}{\Delta x}\right]\right\} \tag{2.48}$$

$$\frac{\partial}{\partial x}\left[\beta^2\left(\frac{\partial u}{\partial z}\right)\right] = \frac{1}{2\Delta x}\left\{(\beta_{n+1,m})^2\left[\frac{u^k_{n+1,m+1} - u^k_{n+1,m-1}}{\Delta z}\right] - (\beta_{n-1,m})^2\left[\frac{u^k_{n-1,m+1} - u^k_{n-1,m-1}}{\Delta z}\right]\right\} \tag{2.49}$$

The homogenous formulation finite-difference scheme of the two-dimensional elastic wave equation for x and z components, can be given as, x-axis

$$\begin{aligned} u_{n,m}^{k+1} =& \gamma^2 \left[u_{n+1,m}^{k} - 2u_{n,m}^{k} + u_{n-1,m}^{k}\right] + \\ & \gamma^2(1-\delta^2)\frac{\left[w_{n+1,m+1}^{k} - w_{n+1,m-1}^{k} + w_{n-1,m+1}^{k} - w_{n-1,m-1}^{k}\right]}{4} + \\ & \gamma^2\delta^2 \left[u_{n,m+1}^{k} - 2u_{n,m}^{k} + u_{n,m-1}^{k}\right] + 2u_{n,m}^{k} - u_{n,m}^{k-1} \end{aligned} \tag{2.50}$$

z-axis

$$\begin{aligned} w_{n,m}^{k+1} =& \gamma^2 \left[w_{n,m+1}^{k} - 2w_{n,m}^{k} + w_{n,m-1}^{k}\right] + \\ & \gamma^2(1-\delta^2)\frac{\left[u_{n+1,m+1}^{k} - u_{n-1,m+1}^{k} + u_{n+1,m-1}^{k} - u_{n-1,m-1}^{k}\right]}{4} + \\ & \gamma^2\delta^2 \left[w_{n+1,m}^{k} - 2w_{n,m}^{k} + w_{n-1,m}^{k}\right] + 2w_{n,m}^{k} - w_{n,m}^{k-1} \end{aligned} \tag{2.51}$$

Where:

$$\gamma = \frac{C_o \Delta t}{h}$$

$$\delta = \frac{\alpha}{\beta}$$

$$u_{n,m}^{k+1} = u(nx, mz, (k+1)\Delta t)$$

$$w_{n,m}^{k+1} = w(nx, mz, (k+1)\Delta t)$$

2.4 Boundary condition techniques

A common problem in finite-difference acoustic and elastic modelling is unwanted reflections from the edge or boundary of a model. The boundary condition of the model is important if we want to decrease unwanted reflections from the boundary. So boundary condition techniques have been developed in many ways (Lysmerr and Kuhlemeyer, 1969; Smith, 1974; Lindman, 1975; Clayton and Engquist, 1977; Reynolds, 1978; Keys, 1985; Vidale and Clayton, 1986; Randall, 1988; Cerjan et al., 1985; Dablain, 1986). In the previous chapter it was noted that boundary condition techniques can be categorized into three methods.

2.4.1 Absorbing boundary condition technique

Reynolds (1978) has developed transparent boundary conditions, which greatly reduce edge reflections. Study of the boundary conditions can be started from a simple 1-D acoustic wave equation as follows,

$$\frac{\partial^2 p}{\partial t^2} = C_o^2 \frac{\partial^2 p}{\partial x^2} \tag{2.52}$$

For $-a \geq x \geq a$, $t \geq 0$ where C_o denotes the velocity of the medium and p is pressure or velocity potential. Wave incidents at boundary conditions $x \pm a$ must pass through these boundaries with no reflection. Therefore the boundary conditions must fulfill the equation,

$$p(+a,t) = p(-a,t) = 0 \tag{2.53}$$

$$\frac{\partial p(+a,t)}{\partial x} = \frac{\partial p(-a,t)}{\partial x} = 0 \tag{2.54}$$

More specifically, suppose that the solution is plane wave propagation given by

$$p = P_o e^{i(\omega t - kx)} \qquad \text{and} \qquad k = \frac{\omega}{C_o} \tag{2.55}$$

Where k is the wave number and ω is the angular frequency, and C_o is the velocity. Then the solution (2.52) for $x \leq a$ is given by

$$p = e^{i(\omega t - kx)} + Re^{i(\omega t + kx)} \tag{2.56}$$

If equation (2.56) is substituted into equations (2.53) and (2.54), it is found that

$$|R| = 1$$

This means that the amplitudes of the reflected waves at $x = a$ and $x = -a$ are the same as the amplitudes of the incident waves. Then equations for the boundary condition may be given as,

$$\frac{1}{C_o}\frac{\partial p(+a,t)}{\partial t} + \frac{\partial p(+a,t)}{\partial x} = 0 \tag{2.57}$$

$$\frac{1}{C_o}\frac{\partial p(-a,t)}{\partial t} - \frac{\partial p(-a,t)}{\partial x} = 0 \tag{2.58}$$

The boundary conditions (2.57) and (2.58) continue by the formal factorization of the differential operator for the wave equation,

$$\frac{1}{C_o}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{C_o}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) - \left(\frac{1}{C_o}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \tag{2.59}$$

And a plane wave $p(x,t)$ which is propagating to the right, must satisfy the following equations,

$$\left(\frac{1}{C_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p = 0 \tag{2.60}$$

And a plane wave $p(x,t)$ which is propagating to the left, must satisfy the equation,

$$\left(\frac{1}{C_o}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)p=0 \tag{2.61}$$

For the case of 1-D or incident waves perpendicular to the edges, it is easy to eliminate reflection from edges. In the 2-D case, the acoustic wave equation is given by,

$$\frac{1}{C_o}\frac{\partial^2 p}{\partial t^2}=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2} \tag{2.62}$$

If we used boundary equations as follows,

$$p(\pm a,z,t)=0, \qquad p(x,b,t)=0, \tag{2.63}$$

$$\frac{p(\pm a,z,t)}{\partial x}=0, \qquad \frac{p(x,b,t)}{\partial z}=0, \tag{2.64}$$

At the boundary sides $(x=\pm a)$, and $(z=b)$ the bottom of the model has strong reflections. If a plane wave propagating to the right is given as,

$$p(x,z,t)=p(x,z,t)=e^{i(\omega t-k_x.x.Cos\theta\pm k_z.z.Sin\theta)}, \qquad 0\le\theta\le\frac{\pi}{2} \tag{2.65}$$

and the plane wave propagating to the left for $x\le a$ is given as,

$$p=e^{i(\omega t-k.x.Cos\theta\pm k.z.Sin\theta)}+R.e^{i(\omega t+k.x.Cos\theta\pm k.z.Sin\theta)} \tag{2.66}$$

Here θ is the angle between the plane wave front and the x-axis, k_x and k_z are the wave numbers. And if equations (2.65)(2.66) are substituted into equations (2.63) and (2.64), we will find that,

$$|R|=1$$

This means that the amplitudes of reflected waves at $x=a$ and $x=-a$ are the same as the amplitudes of the incident waves. Similar to the 1-D case, a factorization of the differential operator of the two-dimensional case may be approximated as,

$$\begin{aligned}\frac{1}{C_o}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}=&\left[\frac{1}{C_o}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial z}\right]\left[\frac{1}{C_o}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right]\\&+2\frac{\partial^2}{\partial x\partial z}\end{aligned} \tag{2.67}$$

$$\frac{1}{C_o}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} = \left[\frac{1}{C_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right]\left[\frac{1}{C_o}\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right] - 2\frac{\partial^2}{\partial x \partial z} \tag{2.68}$$

Suppose that p is a plane wave travelling to the left or the right given by equation (2.65), if the propagation is the horizontal direction, $\theta = 0$, then the equation can be given as,

$$\frac{\partial p}{\partial z} = \frac{\partial^2 p}{\partial x \partial z} \tag{2.69}$$

Thus, the finite-difference scheme for the left side boundary condition may be given as,

$$p^{k+1}_{1,j} = p^k_{1,j} + p^k_{2,j} - p^{k-1}_{2,j} + \frac{C_o \Delta t}{h}\left(p^k_{2,j} - p^k_{1,j} - \left(p^{k-1}_{3,j} - p^{k-1}_{2,j}\right)\right) \tag{2.70}$$

$$2 \le j \le M, \qquad 2 \le k \le T$$

For the right side, the finite-difference scheme is given as,

$$p^{k+1}_{N+1,j} = p^k_{N+1,j} + p^k_{N,j} - p^{k-1}_{N,j} + \frac{C_o \Delta t}{h}\left(p^k_{N,j} - p^k_{N+1,j} - \left(p^{k-1}_{N-1,j} - p^{k-1}_{N,j}\right)\right) \tag{2.71}$$

$$2 \le j \le M, \qquad 2 \le k \le T$$

For the top and bottom boundary conditions, the finite-difference schemes are given as,

$$p^{k+1}_{i,1} = p^k_{i,1} + p^k_{i,2} - p^{k-1}_{i,2} + \frac{C_o \Delta t}{h}\left(p^k_{i,2} - p^k_{i,1} - \left(p^{k-1}_{i,3} - p^{k-1}_{j,2}\right)\right) \tag{2.72}$$

$$p^{k+1}_{i,M+1} = p^k_{i,M+1} + p^k_{i,M} - p^{k-1}_{i,M} + \frac{C_o \Delta t}{h}\left(p^k_{i,M} - p^k_{i,M+1} - \left(p^{k-1}_{i,M-1} - p^{k-1}_{i,M}\right)\right) \tag{2.73}$$

$$2 \le i \le N, \qquad 2 \le k \le T$$

If θ is close to zero, substituting as in equation (2.66) into (2.63) and (2.64), can be solved as,

$$R_1(\theta) = |R| = \frac{1 - Cos\theta}{1 + Cos\theta} \tag{2.74}$$

To derive boundary conditions which reduce the reflection coefficients, the formal factorization of the differential operator for the two-dimensional acoustic wave equation can be considered as,

$$\frac{1}{C_o}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} = \left[\frac{1}{C_o}\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)^{\frac{1}{2}}\right]\left[\frac{1}{C_o}\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)^{\frac{1}{2}}\right] \tag{2.75}$$

$$\text{Let} \qquad L = \sqrt{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)},$$

then,

$$\frac{1}{C_o}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} = \left[\frac{1}{C_o}\frac{\partial}{\partial t} - L\right]\left[\frac{1}{C_o}\frac{\partial}{\partial t} + L\right] \tag{2.76}$$

Based on factorization of equation (2.76) the boundary conditions can be given as,

$$\left(\frac{1}{C_o}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}L\right)p = 0, \qquad x = -a \tag{2.77}$$

$$\left(\frac{1}{C_o}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}L\right)p = 0, \qquad x = a \tag{2.78}$$

The boundary conditions (2.77) and (2.78) can yield a reflection coefficient of zero for all incident angles θ, but explicitly these conditions depend on the wave number, and probably cannot be applied for practical purposes. Wu et al. (1996) considered wave functions to be composed into harmonic components by Fourier transformation. The harmonic function p is defined as,

$$p(x,y,z,t) = P_o sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \tag{2.79}$$

Where ω is the angular frequency, k_x, k_y and k_z are the wave numbers, P_o is the amplitude and ϕ_o is the initial phase. Then the analytical solution of the second partial derivative of equation (2.79) can be written as,

$$\frac{\partial^2 p(x,y,z,t)}{\partial x^2} = -P_o k_x^2 sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \tag{2.80}$$

$$\frac{\partial^2 p(x,y,z,t)}{\partial y^2} = -P_o k_y^2 sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \tag{2.81}$$

$$\frac{\partial^2 p(x,y,z,t)}{\partial z^2} = -P_o k_z^2 sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \tag{2.82}$$

$$\frac{\partial^2 p(x,y,z,t)}{\partial t^2} = -P_o \omega^2 sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \tag{2.83}$$

If acoustic wave propagation is governed by the equation as follows,

$$\frac{\partial^2 p}{\partial t^2} = C_o^2\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right] \tag{2.84}$$

Substituting equations (2.80),(2.81),(2.82),(2.83) into equation (2.84), the dispersion relation can be obtained as,

$$\begin{aligned} -P_o\omega^2 sin(\omega t + k_x x + k_y y + k_z z + \phi_o) =& C_o^2\left(-P_o k_x^2 - P_o k_y^2 - P_o k_z^2\right) \\ & sin(\omega t + k_x x + k_y y + k_z z + \phi_o) \end{aligned} \tag{2.85}$$

$$k^2 = \left(k_x^2 + k_y^2 + k_z^2\right) = \frac{\omega^2}{C_o^2} \tag{2.86}$$

It is sufficient to consider here a side area boundary $x = x_{max}$, and a bottom line boundary $x = x_{max}$, $y = y_{max}$, and $z = z_{max}$. Clayton and Engquist (1977) provide the approach of the absorbing boundary condition for $x = x_{max}$ as follows,

$$\frac{C_o k_x}{\omega} = -1 + O\left(|\frac{C_o k_y}{\omega}|^2 + |\frac{C_o k_z}{\omega}|^2\right), \qquad 2^{nd} \text{ order} \tag{2.87}$$

$$\frac{C_o k_x}{\omega} = -1 + \left(\frac{C_o k_y}{\omega}\right)^2 + \left(\frac{C_o k_z}{\omega}\right)^2 + O\left(|\frac{C_o k_y}{\omega}|^4 + |\frac{C_o k_z}{\omega}|^4\right), \qquad 4^{th} \text{ order} \tag{2.88}$$

Based on the differential form of equation (2.87), the absorbing boundary condition for the side area boundary $x = x_{max}$ can be written as,

$$\frac{\partial^2 p}{\partial x \partial t} + \frac{1}{C_o}\frac{\partial^2 p}{\partial t^2} - \frac{C_o}{2}\frac{\partial^2 p}{\partial y^2} - \frac{C_o}{2}\frac{\partial^2 p}{\partial z^2} \tag{2.89}$$

$$\frac{\partial^2 p}{\partial t^2} = C_o\frac{\partial^2 p}{\partial x \partial t} - \frac{C_o^2}{2}\frac{\partial^2 p}{\partial y^2} - \frac{C_o^2}{2}\frac{\partial^2 p}{\partial z^2} \tag{2.90}$$

In the finite-difference scheme they can be written as,

$$\begin{aligned} p_{i,j,k}^{t+1} =& \frac{C_o dt}{dx}\left[\left(-\frac{3}{2}p_{i,j,k}^{t} + 2p_{i+1,j,k}^{t} - \frac{1}{2}p_{i+2,j,k}^{t}\right) - \right. \\ & \left.\left(-\frac{3}{2}p_{i,j,k}^{t-1} + 2p_{i+1,j,k}^{t-1} - \frac{1}{2}p_{i+2,j,k}^{t-1}\right)\right] + \\ & \frac{1}{2}\left(\frac{C_o dt}{dy}\right)^2\left(p_{i,j+1,k}^{t} - 2p_{i,j,k}^{t} + p_{i,j-1,k}^{t}\right) + \\ & \frac{1}{2}\left(\frac{C_o dt}{dz}\right)^2\left(p_{i,j,k+1}^{t} - 2p_{i,j,k}^{t} + p_{i,j,k-1}^{t}\right) + 2p_{i,j,k}^{t} - p_{i,j,k}^{t-1} \end{aligned} \tag{2.91}$$

$i = 1, N; 2 \leq j \leq M - 1; 2 \leq k \leq L - 1$

Left side boundary($i = 1$),

$$\begin{aligned} p_{1,j,k}^{t+1} =& \frac{C_o dt}{dx}\left[\left(-\frac{3}{2}p_{1,j,k}^{t} + 2p_{2,j,k}^{t} - \frac{1}{2}p_{3,j,k}^{t}\right) - \right. \\ & \left.\left(-\frac{3}{2}p_{1,j,k}^{t-1} + 2p_{2,j,k}^{t-1} - \frac{1}{2}p_{3,j,k}^{t-1}\right)\right] + \\ & \frac{1}{2}\left(\frac{C_o dt}{dy}\right)^2\left(p_{1,j+1,k}^{t} - 2p_{1,j,k}^{t} + p_{1,j-1,k}^{t}\right) + \\ & \frac{1}{2}\left(\frac{C_o dt}{dz}\right)^2\left(p_{1,j,k+1}^{t} - 2p_{1,j,k}^{t} + p_{1,j,k-1}^{t}\right) + 2p_{1,j,k}^{t} - p_{1,j,k}^{t-1} \end{aligned} \tag{2.92}$$

$2 \leq j \leq M-1; 2 \leq k \leq L-1$

Right side boundary($i = N$),

$$\begin{aligned}p^{t+1}_{N,j,k} =& \frac{C_o dt}{dx}\left[\left(-\frac{3}{2}p^t_{N,j,k} + 2p^t_{N-1,j,k} - \frac{1}{2}p^t_{N-2,j,k}\right) - \right.\\ &\left.\left(-\frac{3}{2}p^{t-1}_{N,j,k} + 2p^{t-1}_{N-1,j,k} - \frac{1}{2}p^{t-1}_{N-2,j,k}\right)\right] + \\ &\frac{1}{2}\left(\frac{C_o dt}{dy}\right)^2 \left(p^t_{N,j+1,k} - 2p^t_{N,j,k} + p^t_{N,j-1,k}\right) + \\ &\frac{1}{2}\left(\frac{C_o dt}{dz}\right)^2 \left(p^t_{N,j,k+1} - 2p^t_{N,j,k} + p^t_{N,j,k-1}\right) + 2p^t_{N,j,k} - p^{t-1}_{N,j,k}\end{aligned} \tag{2.93}$$

$2 \leq j \leq M-1; 2 \leq k \leq L-1$

The absorbing boundary condition for the side area boundary $z = z_{max}$ can be written as,

Top side boundary ($k = 1$),

$$\begin{aligned}p^{t+1}_{i,j,1} =& \frac{C_o dt}{dz}\left[\left(-\frac{3}{2}p^t_{i,j,1} + 2p^t_{i,j,2} - \frac{1}{2}p^t_{i,j,3}\right) - \right.\\ &\left.\left(-\frac{3}{2}p^{t-1}_{i,j,1} + 2p^{t-1}_{i,j,2} - \frac{1}{2}p^{t-1}_{i,j,3}\right)\right] + \\ &\frac{1}{2}\left(\frac{C_o dt}{dx}\right)^2 \left(p^t_{i+1,j,1} - 2p^t_{i,j,1} + p^t_{i-1,j,1}\right) + \\ &\frac{1}{2}\left(\frac{C_o dt}{dy}\right)^2 \left(p^t_{i,j+1,1} - 2p^t_{i,j,1} + p^t_{i,j-1,1}\right) + 2p^t_{i,j,1} - p^{t-1}_{i,j,1}\end{aligned} \tag{2.94}$$

$2 \leq i \leq N-1; 2 \leq j \leq M-1$

Bottom side boundary ($k = L$),

$$\begin{aligned}p^{t+1}_{i,j,L} =& \frac{C_o dt}{dz}\left[\left(-\frac{3}{2}p^t_{i,j,L} + 2p^t_{i,j,L-1} - \frac{1}{2}p^t_{i,j,L-2}\right) - \right.\\ &\left.\left(-\frac{3}{2}p^{t-1}_{i,j,L} + 2p^{t-1}_{i,j,L-1} - \frac{1}{2}p^{t-1}_{i,j,L-2}\right)\right] + \\ &\frac{1}{2}\left(\frac{C_o dt}{dx}\right)^2 \left(p^t_{i+1,j,L} - 2p^t_{i,j,L} + p^t_{i-1,j,L}\right) + \\ &\frac{1}{2}\left(\frac{C_o dt}{dy}\right)^2 \left(p^t_{i,j+1,L} - 2p^t_{i,j,L} + p^t_{i,j-1,L}\right) + 2p^t_{i,j,L} - p^{t-1}_{i,j,L}\end{aligned} \tag{2.95}$$

$2 \leq i \leq N-1; 2 \leq j \leq M-1$

The absorbing boundary condition for the side area boundary $y = y_{max}$ can be written as, Left side boundary in axis ($j = 1$),

$$\begin{aligned} p^{t+1}_{i,1,k} =& \frac{C_o dt}{dy}\left[\left(-\frac{3}{2}p^{t}_{i,1,k} + 2p^{t}_{i,2,k} - \frac{1}{2}p^{t}_{i,3,k}\right) - \right. \\ & \left.\left(-\frac{3}{2}p^{t-1}_{i,1,k} + 2p^{t-1}_{i,2,k} - \frac{1}{2}p^{t-1}_{i,3,k}\right)\right] + \\ & \frac{1}{2}\left(\frac{C_o dt}{dx}\right)^2 \left(p^{t}_{i+1,1,k} - 2p^{t}_{i,1,k} + p^{t}_{i-1,1,k}\right) + \\ & \frac{1}{2}\left(\frac{C_o dt}{dz}\right)^2 \left(p^{t}_{i,1,k+1} - 2p^{t}_{i,1,k} + p^{t}_{i,1,k-1}\right) + 2p^{t}_{i,1,k} - p^{t-1}_{i,1,k} \end{aligned} \tag{2.96}$$

$2 \leq i \leq N-1; 2 \leq k \leq L-1$

Right side boundary in axis ($k = 1$),

$$\begin{aligned} p^{t+1}_{i,M,k} =& \frac{C_o dt}{dy}\left[\left(-\frac{3}{2}p^{t}_{i,M,k} + 2p^{t}_{i,M-1,k} - \frac{1}{2}p^{t}_{i,M-2,k}\right) - \right. \\ & \left.\left(-\frac{3}{2}p^{t-1}_{i,M,k} + 2p^{t-1}_{i,M-1,k} - \frac{1}{2}p^{t-1}_{i,M-2,k}\right)\right] + \\ & \frac{1}{2}\left(\frac{C_o dt}{dx}\right)^2 \left(p^{t}_{i+1,M,k} - 2p^{t}_{i,M,k} + p^{t}_{i-1,M,k}\right) + \\ & \frac{1}{2}\left(\frac{C_o dt}{dz}\right)^2 \left(p^{t}_{i,M,k+1} - 2p^{t}_{i,M,k} + p^{t}_{i,M,k-1}\right) + 2p^{t}_{i,M,k} - p^{t-1}_{i,M,k} \end{aligned} \tag{2.97}$$

$2 \leq i \leq N-1; 2 \leq k \leq L-1$

2.4.2 Non-reflecting boundary condition techniques

Absorption techniques or non-reflecting boundary condition proposed for the finite-element and finite-difference methods (Lysmerr and Kuhlemeyer, 1969; Smith, 1974; Lindman, 1975; Clayton and Engquist, 1977; Reynolds, 1978; Keys, 1985; Vidale and Clayton, 1986; Randall, 1988) are relatively successful. But these methods are still not effective for outward wave propagation on the boundary at shallow angles. Difficulties in applying the boundary condition methods also increase for complex geometry environments. Therefore some researchers have attempted to develop additional methods to reduce non-physical reflections from boundaries. Cerjan et al. (1985) illustrated the nonreflecting boundary condition for the Fourier method for solving Kosloff and Baysal's (1982) acoustic wave equation. If $P^n(i,j)$ denotes the pressure at time $t = nDt$ and at spatial location $x = iDx$, $y = jDy$, $i = 1...N_x$, $j = 1...N_y$, where dt denotes the time step size Dx and Dy denote the mesh size in the $x-$ and $y-$ directions. A typical time step of the solution method runs as follows,

1. Calculate

$$R^n(i,j) = \frac{\partial^2 P^n}{\partial x^2} + \frac{\partial^2 P^n}{\partial y^2} \tag{2.98}$$

   by the Fourier approximation

2. integrate in time according to,

$$\begin{aligned} P^{n+\frac{1}{2}}(i,j) &= P^{n-\frac{1}{2}}(i,j) + DtR^n(i,j)C^2(i,j) \\ P^{n+1}(i,j) &= P^n(i,j) + DtP^{n+\frac{1}{2}} \end{aligned} \tag{2.99}$$

Where $C(i,j)$ denotes the acoustic velocity. The calculations in step (1) and step (2) are repeated for the desired number of time steps. For the nonreflecting boundary condition the values of amplitudes are slightly reduced after each time step in a strip of grid points surrounding the numerical mesh. For the Fourier method a strip width of twenty grid points was found sufficient to reduce side reflections to a few percent. The amplitude reduction in each strip is gradually tapered from a zero value in the interior boundary. For their example results, the pressure amplitudes are multiplied by the factor G where,

$$G = \exp^{-[\beta(20-i)]^2} \tag{2.100}$$

where the β is a tapering factor. In their application Cerjan, Kosloff, Kosloff and Reshef use $\beta = 0.0015$. i gives a value of 1 for $i = 20$ and a value of 0.92 for $i = 1$. For the lower strip a width of 40 is used with maximum amplitude reduction in the centre for the strip. For free surface problems the lower strip width is 20 with maximum reduction on the boundary. For elastic wave calculations the nonreflecting boundary conditions can be affected by reducing the magnitudes of the field variables at the end of each time step. For the stress equation method (Kosloff and Baysal, 1982) the variables include the stress components and the stress-time derivatives.

2.4.3 Transmissive sponge boundary condition techniques

To reduce non-physical reflections from boundaries, Dablain (1986) implemented transmissive sponge boundaries for higher order difference schemes. The idea of this technique originally is that of Israeli and Orszag (1981). Israeli and Orszag used this technique to approximate radiation conditions at numerical boundaries. The transmission operator is applied in such a way that it supports radiation in one direction and damps it in the opposite, thereby attenuating any residual boundary reflections. The simplest operator designed to implement this idea is written in one dimension as,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} - \frac{\epsilon}{c}\left(\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial x}\right) \qquad \text{with} \qquad \epsilon = 0 \tag{2.101}$$

Intuition suggests that when the wave propagation is to the right, $\phi(x-ct)$, then $\frac{\partial\phi}{\partial t}+c\frac{\partial\phi}{\partial x}$ vanishes and the wave equation is left intact. When the wave propagation to the left, however, then $\phi = \phi(x+ct)$ and

$$\frac{\partial\phi}{\partial t} = c\frac{\partial\phi}{\partial\xi}; \qquad \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}; \qquad \text{Where: } \xi = x + ct \tag{2.102}$$

leaving an error term of the form $2c\frac{\partial\phi}{\partial x}$.

The effect of adding a first-order derivative to the wave equation is that of a damping term. The result then of the equation is to transmit waves in one direction and attenuate them in the other. If the right-hand side of the grid is considered, one form of the sponge boundary condition would solve again as,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} - \epsilon(x)\left(\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial x}\right) \tag{2.103}$$

The damping coefficient should go from zero for the interior of the mesh, to its maximum value at the right-hand boundary. A simple linear ramp over twenty points with a flat top over ten points was used in this study. The above equation suggests that the sponge boundary term can be applied after the time update to the mesh has already taken place. If it is assumed the update has already taken place, the sponge damping in the right-hand boundary layer would appear as,

$$\widetilde{\phi}_i^{n+1} = \phi_i^{n+1} - \varepsilon_i\left[\phi_i^{n+1} - \phi_i^n + \frac{c\Delta t}{\Delta x}\left(\phi_{i+1}^n - \phi_i^n\right)\right] \tag{2.104}$$

where: ϕ_i^{n+1} is a time updated mesh point, $\widetilde{\phi}_i^{n+1}$ is a time updated mesh point with boundary reflections damped, and ε_i is the sponge damping coefficient or reduction factor in the boundary zone.

Two additional points should be made with regard to this sponge layer. The first point concerns the value of ε_i. The maximum value used in any numerical experiments has been 0.1. This value was used in combination with a Courant number (Cr) of 0.5. The Courant number is defined as $Cr = \frac{c\Delta t}{\Delta x}$. Heuristic reasoning suggests that any perturbing term added to the wave equation should be small in relationship to the physically important parameters in the zone of interest. The physically significant parameter in this case is the Courant number. This reasoning suggests that the sponge layer parameter should be adjusted up or down in relation to the Courant number for the scheme, and its proximity to the phenomenon of interest. The second point concerns the effectiveness of the sponge layer for a fixed number of boundary points. The capacity of the sponge layer for damping is a function of how many wave lengths are traversed in that zone. This means that for a fixed number of grid cells the boundary layer becomes more effective if coarse-grid sampling can be used. The practical consequence is that a coarse-grid scheme will more effectively damp grid reflections that a fine-grid scheme for the same number of boundary points, in a sponge-zone formulation.

2.5 Viscous and heat conduction absorption of acoustic waves

Viscous and heat conduction losses are two types of losses which dominate the absorption of sound in the ocean. Processes of relative motion between adjacent portions of the medium, compressions and expansions in the transmission of a sound wave, which result in viscous losses lead to viscous absorption. Processes of the conduction of thermal energy in the water medium, which result in heat losses, are known as heat conduction absorption. The absorption processes of the transmission of sound by using the viscosity were introduced by Stokes, who developed the first successful theory of sound absorption. An additional contribution for sound absorption process in the medium utilized by Kirchhoff, who introduced modified absorption processes linked to the property of thermal conductivity. Two theories of viscous and heat conduction absorption constitute what is generally called classical sound absorption in fluids. It is known that a linear acoustic wave equation of the sound propagation in fluids can be presented as,

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{2.105}$$

Where p is the acoustic pressure and c is the phase speed for acoustic waves in fluids. The phase speed c is defined as,

$$c=\sqrt{\frac{K}{\rho_o}} \tag{2.106}$$

Where K is the bulk modulus, the acoustic pressure p in the acoustic wave equation in Equation (2.105) can be written as,

$$p=\rho_o c^2 s \tag{2.107}$$

Where s is the condensation and ρ_o is the density, and in order to satisfy the wave equation, s must be proportional to the pressure p. To introduce energy losses in acoustic wave modelling there is one known way which to modify the equation of state to allow for a delay between the application of sudden pressure change and the attainment of the resulting equilibrium condensation given by the above equation. Stokes considered the first simple relation of a modified equation of state as follows,

$$p=\rho_o.c^2\left(1+\tau\frac{\partial}{\partial t}\right)s \tag{2.108}$$

Where τ is a delay time, which is commonly known as a relaxation time, and each of the absorption processes is characterized by the relaxation time, which measures the amount of time for the particular process to be nearly completed. So the solution for the condensation s satisfies this condition,

$$s=\frac{P_o}{\rho_o.c^2}(1-e^{-t/\tau}) \tag{2.109}$$

Continuing this relaxation theory, a modified wave equation can be obtained as a modified three-dimensional acoustic wave equation as follows,

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}&=\left(1+\tau\frac{\partial}{\partial t}\right)\nabla^2 p\\ &=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}+\tau\left(\frac{\partial^3 p}{\partial x^2\partial t}+\frac{\partial^3 p}{\partial y^2\partial t}+\frac{\partial^3 p}{\partial z^2\partial t}\right)\end{aligned} \tag{2.110}$$

As a standard to describe sound absorption in the fluids, an absorption coefficient α is introduced. Generally this coefficient is known as classical absorption coefficient, presented as the sum of absorption coefficient of individual loss mechanisms calculated as,

$$\alpha = \sum_i \alpha_i \tag{2.111}$$

The absorption coefficient α, which is produced by viscous loss and thermal conduction mechanisms, can be presented as,

$$\begin{aligned} \alpha =& \alpha_s + \alpha_\kappa \\ \alpha =& \left(\frac{2}{3}\frac{\omega^2\eta}{\rho_0 c^3}\right) + \left(\frac{1}{2}\frac{\omega^2\kappa(\gamma-1)}{\rho_0 c^3}\right) \end{aligned} \tag{2.112}$$

If the viscosity relaxation and the heat conduction relaxation is given as,

$$\tau_{viscosity} = \tfrac{2c}{\omega^2}\alpha_s \tag{2.113}$$

$$\tau_{heat\,conduction} = \tfrac{2c}{\omega^2(\gamma-1)}\alpha_\kappa \tag{2.114}$$

Equation (2.112) also may be arranged in the form of the relaxation time τ as follows,

$$\begin{aligned} \tau =& \tau_{viscosity} + \tau_{heat\,conduction} \\ =& \left(\frac{2c}{\omega^2}\alpha_s\right) + \left(\frac{2c}{\omega^2(\gamma-1)}\alpha_\kappa\right) \\ =& \frac{2c}{\omega^2}\left(\alpha_s + \frac{1}{(\gamma-1)}\alpha_\kappa\right) \end{aligned} \tag{2.115}$$

Where γ is the ratio of heat capacities $C_\Re/C_\nu$. C_ν is the heat capacity and $C_\Re$ is the heat capacity at constant pressure. Usually in water and seawater the absorption produced by thermal conductivity is negligible compared with that from viscosity (Kinsler, Frey, Coppens and Sanders, 1982). So the 2-D acoustic wave modelling in fluid media medium with loss mechanisms may be presented as follows,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + \tau\frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2}\right) \tag{2.116}$$

This modified acoustic wave equation is used for acoustic wave modelling in homogeneous media. For heterogeneous media the equation can be written as,

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} =& \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \\ & \tau\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right)\right\} \end{aligned} \tag{2.117}$$

Where:

$\tau = \frac{2c}{\omega^2}\alpha =$ the relaxation time (second)

$\alpha =$ the absorption coefficient (Np/meter)

$c =$ the sound speed (meter/second)

$\omega = 2.\pi.f =$ the angular frequency (Hz)

$f =$ the frequency of sound (Hz)

Equations (2.116) and (2.117) are adopted here to study an acoustic wave propagation in underwater media involving attenuation characteristics. The seawater acoustic absorption adopted here, though, does not incorporate molecular processes associated with Boric Acid ($B(OH)_3$) at lower frequencies and Magnesium Sulphate ($MgSO_4$) at higher frequencies. Nevertheless, the contribution of the acoustic seawater absorption at 20 Hz for a range of 3 km ($1.8x10^{-3}$ dB) as discussed in Chapter 8 on the total transmission loss is very small in comparison to that of the spherical spreading effect (69.54 dB). The inclusion of molecular processes, however, is best considered for future work. In the current research, the equations utilising classical absorption only are used to simulate acoustic wave propagation in the ocean.

2.6 Transmission Loss

The energy of sound wave propagation in the medium will dissipate with range. The sources of dissipation are divided into two general categories: those due to losses in the medium and losses at the boundaries of the medium. The first is of most importance when the volume of the fluid is large, as in the transmission of sound in the Earth's ocean. The second becomes important when significant interaction with system boundaries notably the seabed, is involved. (Kinsler and Frey, 1962; Kinsler et al., 1982). Losses of the energy in the medium may be also divided into three basic types: viscous energy losses, heat energy losses and molecular exchanging energy losses.

In the Earth's ocean, the sea and its boundaries form a remarkably complex medium for the propagation of sound (Urick, 1967). It can affect an acoustic signal travelling through the ocean because the signal is subject to geometric spreading, distorted by multipath effects and weakened due to various loss mechanisms. The standard measure in underwater acoustics of the change in signal strength with range is Transmission Loss (Jensen et al., 1994). Transmission loss is defined as the ratio in decibels between the acoustic intensity $I(r,z)$ at a field point and intensity I_o at 1m distance (r) from the source (Jensen et al., 1994). The transmission loss may be presented as follows,

$$\begin{aligned} \text{Transmission Loss } (TL) &= -10.\log\frac{I(r,z)}{I_o} \qquad [\text{dB}] \qquad (2.118)\\ \text{Transmission Loss } (TL) &= -20.\log\frac{p(r,z)}{p_o} \qquad [\text{dB}] \qquad (2.119)\end{aligned}$$

Transmission loss can be calculated to be the sum of a loss due to geometrical spreading and a loss due to attenuation (Jensen et al., 1994). The two geometries of importance for studying spreading loss in underwater acoustics are spherical and cylindrical spreading geometries. The spreading loss is defined as a measure of signal weakening of sound propagation that propagates outward from the source to the medium. The Transmission Loss due to spherical spreading may be calculated as,

$$\begin{aligned} \text{Spherical spreading:} \qquad TL &= -\,10.\log\frac{I_s}{I_o}\\ TL &= -\,10.\log\frac{\left(\dfrac{P}{4.\pi.r^2}\right)}{\left(\dfrac{P}{4.\pi.1^2}\right)} \qquad (2.120)\\ TL &= -\,10.\log r^{-2}\\ TL &= 20.\log r \qquad [\text{dB}]\end{aligned}$$

For a point source in a waveguide, we have spherical spreading in the nearfield ($r \leq D$). For cylindrical spreading the Transmission Loss may be calculated as,

$$\begin{aligned} \text{Cylindrical spreading:} \qquad TL &= -\,10.\log\frac{I_c}{I_o}\\ TL &= -\,10.\log\frac{\left(\dfrac{P}{4.\pi.r.1}\right)}{\left(\dfrac{P}{4.\pi.1.1}\right)} \qquad (2.121)\\ TL &= 10.\log r \qquad [\text{dB}]\end{aligned}$$

This spreading is applied at longer range in the farfield. The intensity of a sound wave also depends on characteristics of the medium. In a water medium with density ρ and sound speed C, the average intensity I of a plane wave with the root mean squares of the pressure p, may be presented as shown in Equation (2.122)(Jensen et al., 1994),

$$I = \frac{p^2}{\rho C} \tag{2.122}$$

The TL can be calculated by substituting Equation (2.122) into Equation (2.118) as given in Equation (2.123),

$$TL = -10 \log \left(\frac{\displaystyle\sum_{t=0}^{n} \frac{(p(r,z,t))^2}{\rho(r,z).C(r,z)}}{\displaystyle\sum_{t=0}^{n} \frac{(p(r_s,z_s,t))^2}{\rho(r_s,z_s).C(r_s,z_s)}} \right) \tag{2.123}$$

In the modelling techniques of acoustic wave propagation, Equation (2.123) is used to calculate th propagation loss of acoustic waves and was adopted here in this study. The pressure p at the receiver given as $p(r, z, t)$ in Equation (2.123) adopted here has included the absorption coefficient as shown as the relaxation time τ in Equations (2.116) and (2.117).

2.7 2.5-D wave modelling theory

Commonly, in studying acoustic and elastic wave propagation in underwater or under seabed sediments, the characteristics of the media in wave propagation environments are treated as having variations in 2-D only. This can give an impression that to analyze wave propagation signals, 3-D acoustic and elastic wave modelling is not as important as 2-D modelling. Also, 2-D modelling needs less CPU time and computer memory than 3-D modelling. But 3-D acoustic and elastic wave modelling give real and true wave propagation effects and meaningful results. This is a significant issue, increasing the attention of researchers to develop acoustic and elastic wave propagation modelling based on 2-D analysis that represent the effects of real environments.

The first paper introducing a mathematical formulation of ray theory of the so-called two and a half-dimensional (2.5-D) wave propagation modelling technique, was presented by Bleistein (1986). He used two and a half-dimensional plane wave propagation to model a 3-D wave propagation in the frequency domain that has variation in 2-D environments and for constant and variable density media. His results of interest were for sources and receivers at a plane with parameter variations allowed in two directions. He used Green's functions and the Kirchhoff approximation to analyse modification of 2-D wave propagation to simulate three dimensional spreading. This method was applied to approximate the upward scattered field from a single reflector. It was found that the in-plane propagation of a wave in three dimensions could be described totally.

While ray theory is useful and versatile at high frequencies, there are still many low frequency effects such as Fresnel zones, wave tunnelling, etc that are not represented by ray theory. Ray theory cannot be used to compute a complete wave-field containing every possible arrival, because ray tracing requires individual events which should be specified from the computed wavefield. Therefore, a method commonly used to treat low frequency waves is time domain finite-difference modelling. It is based directly on the wave equation. A different approach was followed by Liner (1991) who developed the 2.5-D acoustic wave equation for constant density media using a 3-dimensional Green's function. One of the advantages of using the wave equation over the ray-based equation, is that all wave species are automatically represented in the wave equation simultaneously. 2.5-D acoustic wave equations for constant and variable density media based on a WKBJ series analysis of Liner's (1991) equation are derived by Stockwell (1995).

Williamson and Pratt (1995) proposed an operator to determine the 2.5-D response by simply applying the filter F to the 2-D response for correcting amplitudes and adjusting phases of 2-D to 2.5-D solutions. This is because of asymptotic similarity of the kinematics of the 2-D and 3-D problems. However they doubt that any operator exists that can be applied to 2.5-D traces in arbitrary velocity fields, to give an exact conversion without event identification and ray tracing, because $\sigma(X,t)$ in Equation (2.125) below is potentially multivalued. $\sigma(X,t)$ is a spatially and temporally varying damping coefficient which is a function of total traveltime and wave velocity. In the present numerical modelling scheme, $\sigma(X,t)$ is modified at each time step and grid point. Liner (1991) proposed a partial differential equation based on an in-plane representation of the

3-D Green's function, and suggested it would simulate the 2.5-D acoustic wave response in a 2-dimensional constant density media. (Liner, 1991)'s 2.5-D wave equation for a variable wave speed and constant density media is,

$$\left[\nabla^2 - \frac{1}{c^2(X)}\left\{\frac{\partial^2}{\partial t^2} + \frac{1}{T}\frac{\partial}{\partial t} + \frac{1}{T^2}\right\}\right] P(X,t) = 0 \tag{2.124}$$

Where $c(X)$ is the wave velocity as function of the x and z coordinates, and T is the total traveltime. After Liner, Stockwell (1995) found that the last terms of Equation (2.124) may be negligible. Therefore Stockwell (1995) proposed a 2.5-D acoustic wave equation for constant density media,

$$\left[\nabla^2 - \frac{1}{c^2(X)}\frac{\partial^2}{\partial t^2} - \frac{1}{\sigma(X,t)}\frac{\partial}{\partial t}\right] P(X,t) = 0 \tag{2.125}$$

For variable density media, Stockwell (1995) also proposed a 2.5-D acoustic wave equation as follows,

$$\rho(X)\nabla\left(\frac{\nabla P(X,t)}{\rho(X)}\right) - \frac{1}{\sigma(X,t)}\frac{\partial P(x,t)}{\partial t} - \frac{1}{c^2(X)}\frac{\partial^2 P(X,t)}{\partial t^2} = 0 \tag{2.126}$$

Where $\sigma(X,t) = c^2(X)T$ is a function of total traveltime and wave velocity. In the same year, Williamson and Pratt (1995) proposed a similar 2.5-D acoustic wave propagation as follows,

$$\hat{F}\left[\nabla^2 - \frac{1}{c^2(X)}\left\{\frac{\partial^2}{\partial t^2} + \frac{1}{T}\frac{\partial}{\partial t} - \frac{1}{4T^2}\right\}\right] P(X,t) = S(x,z;t) \tag{2.127}$$

Where the $\hat{F}$ is a filter to approximate the 2.5-D constant density acoustic wave equation. Williamson and Pratt (1995) derived the filter $\hat{F}$ based on Bleistein's (1986) consideration of the 2.5-D problem in the context of ray theory to obtain the conversion as in Equation (2.128),

$$\widetilde{\psi}^{(2.5)}(\omega) = \widetilde{\psi}^{(2)}(\omega)\sqrt{\frac{|\omega|}{2\pi\sigma}}\exp\{-i\pi/4 \text{ sgn } \omega\} = \widetilde{\psi}^{(2)}(\omega)\sqrt{\frac{-i\omega}{2\pi\sigma}} \tag{2.128}$$

Equation(2.128) can be presented in the form of Equation (2.129)

$$\widetilde{\psi}^{(2)}(\omega) = \widetilde{\psi}^{(2.5)}(\omega)\sqrt{\frac{2\pi\sigma}{|\omega|}}\exp\{i\pi/4 \text{ sgn } \omega\} = \widetilde{\psi}^{(2.5)}(\omega)\sqrt{\frac{2\pi\sigma}{-i\omega}} \tag{2.129}$$

Where $\widetilde{\psi}^{(n)}(\omega)$ is the component at angular frequency ω of the acoustic wavefield in n dimensions, and ψ is the ray parameter. If $d\psi = cds = c^2 d\tau$ and where s and

τ are the distance and traveltime t along the ray, respectively. The term $\sqrt{2\pi\sigma}$ may be presented as,

$$\sqrt{2\pi\sigma} = c_o\sqrt{2\pi t}$$

From Equation (2.129), the $(-i\omega^{-\frac{1}{2}})$ amounts to a half-integrator $D_{-1/2}$. The $D_{-1/2}$ is a half-integration with respect to time and the t is traveltime. Then Williamson and Pratt (1995) approximate $\hat{F}$ by,

$$\hat{F} = D_{-1/2}.c_o\sqrt{2\pi t} \tag{2.130}$$

Deregowski and Brown (1983) proposed the half-integrator $D_{-1/2}$ can be achieved in the time domain by convolution with an equation as follows,

$$D_{-1/2}(t) = \frac{H(t)}{\sqrt{(\pi t)}} \tag{2.131}$$

Where $H(t)$ is the heaviside step function or full integrator (Deregowski and Brown, 1983). Note that the unit Heaviside step function $H(t)$ can be defined by,

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{2.132}$$

Then solution of $D_{-1/2}(t)$ may be presented as,

$$D_{-1/2}(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\sqrt{(\pi t)}} & t > 0 \end{cases} \tag{2.133}$$

For $t > 0$ filter operator $\hat{F}$ may be presented as,

$$\hat{F} = \frac{1}{\sqrt{\pi t}} c_o\sqrt{2\pi t} = c_o\sqrt{2} \tag{2.134}$$

Factor $\hat{F}$ is an additional stage and represents the main difference between Equation (2.127) and Liner's (1991) equation. In Equation (2.127) Williamson and Pratt (1995) use the filter operator $\hat{F}$ to approximate the 2-D model by a 2.5-dimensional model. A filter operator used to approximate 2.5-dimensional model by 2-dimensional model can presented as $\hat{G}$. From Equation (2.128), the filter operator $\hat{G}$ can be derived as follows,

$$\hat{G} = \sqrt{\frac{-i\omega}{2\pi\sigma}} \tag{2.135}$$

Where $(-i\omega)^{1/2}$ amounts to half-derivative $D_{1/2}$. Following Williamson and Pratt (1995) the filter operator $\hat{G}$ may be approximated by,

$$\hat{G} = D_{1/2}.\frac{1}{\sqrt{2\pi\sigma}} = D_{1/2}.\frac{1}{c_o\sqrt{2\pi t}} \tag{2.136}$$

Deregowski and Brown (1983) proposed the half-derivative $D_{1/2}$ also can be achieved in the time domain by convolution with an equation as follows,

$$D_{1/2}(t) = \frac{1}{\sqrt{\pi}}\Big(\frac{\delta(t)}{\sqrt{t}} - \frac{H(t)}{2t^{3/2}}\Big) \tag{2.137}$$

If Equation (2.132) is substituted into Equation (2.137), the above equation can be presented as,

$$D_{1/2}(t) = \frac{1}{\sqrt{\pi}}\Big(\frac{\delta(t)}{\sqrt{t}} - \frac{1}{2t^{3/2}}\Big) \tag{2.138}$$

Then the filter operator $\hat{G}$ can be presented as,

$$\begin{aligned}\hat{G} &= \frac{1}{\sqrt{\pi}}\Big(\frac{\delta(t)}{\sqrt{t}} - \frac{1}{2t^{3/2}}\Big)\frac{1}{c_o\sqrt{2\pi t}} \\ &= \frac{1}{c_o\pi t\sqrt{2}}\Big(\delta(t) - \frac{1}{2t}\Big)\end{aligned} \tag{2.139}$$

The filter operator $\hat{G}$ is similar to the filter operator $\hat{F}$. The filter operator $\hat{F}$ is used to approximate 2-D by 2.5-D, and the filter operator $\hat{G}$ is used to approximate 2.5-D by 2-D. Unfortunately, Williamson and Pratt (1995) did not apply the theoretical formula to simulate 2.5-D acoustic wave propagation. Therefore this theoretical approach still needs to be studied in order to be able to be applied to 2.5-D acoustic wave propagation modelling. Before the work of Williamson and Pratt, the theory and application of the filter operator were studied by Esmersoy and Oristaglo (1988). They proposed the filter operator as a "prefilter". Their formula is presented as follows,

$$\hat{F} = w(t) = \sqrt{\frac{2|t|}{\pi}} \tag{2.140}$$

Where $|t|$ is the travel time. Esmersoy and Oristaglo (1988) don't include the wavespeed c_o in their formula or assume that the wavespeed is constant everywhere. Actually this formulation is not quite true if it is assumed that the formulation of the filter operator does not affect the wavespeed. But the formula has been applied in modelling 2.5-D acoustic wave propagation. Therefore Esmersoy and Oristaglo's (1988) filter operator is still applicable to model and simulate

2.5-D time domain acoustic wave propagation. The term $|t|$ in Equation (2.140) may be expressed as,

$$|t| = \Delta t.L \tag{2.141}$$

Here $L = 1, 2, 3...N$ and Δt is the time step. If Equation (2.141) is substituted into Equation (2.140), we can find,

$$\hat{F} = K\sqrt{L} \tag{2.142}$$

Where K is equal to $\sqrt{\frac{2\Delta t}{\pi}}$ as a constant of filter operator $\hat{F}$. Equation (2.142) is a basic equation which will be used to study filter terms of the 2.5-D acoustic modelling in this research. Narayan (1999) has studied the development of a 2.5-D simulation technique by applying a 2.5-D model in media with variable density and velocity. In this study Narayan (1999) used a 2.5-D acoustic model proposed by Stockwell, to study the effect of heterogeneity on the amplitude and shape of the wave front, by using snapshots in a crosshole geometry, calculated at various times.

In his study Narayan also presented comparisons of modelled 2-D and 2.5-D acoustic responses for variable and constant density in simple two layer media. The results indicated that 2.5-D modelling techniques give pressure amplitudes of acoustic energy spreading less than amplitudes of 2-D acoustic model. He also did not involve terms of filter operator in his algorithm. Narayan did not study simulation of 3-D acoustic modelling, so he could not compare the results with 3-D modelling results.

So far application of 2.5-D acoustic equation has been discussed, but several problems associated with the 2.5-D acoustic wave equation have not been resolved. Does the equation satisfy the requirements of spherical spreading of energy? How far can the equation be used to approximate real acoustic wave propagation in three-dimensional environments?. Can this technique be used for modelling elastic wave propagation? These equations are addressed in the present work.

2.8 Application of FFT in numerical modelling

Originally the application of the Fast Fourier transform in numerical modelling was proposed by Kreiss and Oliger (1972). After Kreiss and Oliger, Gazdag (1981) also proposed a Fourier transform method to provide solutions of 2-D acoustic wave modelling, which can give higher accuracy results than by using finite-difference formulas. Based on equation (2.8), a 1-D acoustic wave equation for homogeneous formulation can be presented as follows,

$$\frac{\partial^2 p}{\partial t^2} = C_o^2 . \left[\frac{\partial^2 p}{\partial x^2} \right] \tag{2.143}$$

To solve the Fourier transform solution of the second derivative of $p(x,t)$ with respect to x, the Fourier transform solution of Equation (2.143) can be obtained in two steps. In the the first step, $p(x_n,t)$ is transformed into $P(k_m,t)$ as follows,

$$P(k_m,t) = \frac{\Delta x}{2\sqrt{\pi}} \sum_{n=-N/2}^{n=N/2} p(x_n, t=0).e^{\frac{-2\pi i}{M}.m.n} \tag{2.144}$$

The second step, to find the second derivative of $p(x,t)$, $P(k,t)$ should be multiplied by $-k_m^2$ and inverted by using the inverse Fourier transform as follows,

$$\frac{\partial^2 p}{\partial x^2} = \frac{\sqrt{2\pi}}{\Delta x M} \sum_{m=-M/2}^{m=M/2} \left[(-k_m^2).P(k_m, t=0) \right] .e^{\frac{+2\pi i}{M}.m.n} + O(h^\infty) \tag{2.145}$$

Following the second temporal derivative in Equation (2.30), the second spatial derivative of $p(x,t)$ can be approximated by using a standard central difference equation in second order accuracy as,

$$\frac{\partial^2 p}{\partial x^2} = \frac{p_{n+1}^k - 2.p_n^k + p_{n-1}^k}{(\Delta x)^2} + O(h^2) \tag{2.146}$$

Where $p_n^k = p(n\Delta x, k\Delta t)$, Δx represent the distance between the horizontal grid points, and Δt is the time step increment.

Approximation of the second spatial derivative of $p(k,t)$ by using the Fourier transform method as shown in Equation (2.145) is more accurate if it is compared with the finite-difference method as shown in Equation (2.146), because truncated errors of approximation in Equation (2.145) are no error since $h^\infty \to 0$, if compared with truncated errors in Equation (2.146), or $O(h^\infty) \lll O(h^2)$.

Commonly, terms of the second temporal derivative of $p(x,t)$ still uses a standard central difference equation as shown in Equation (2.30), in order to simulate wave propagation in time series. This Fourier transform method is better known as the Pseudospectral method.

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